Two-point Boundary Value Problems

1. Let $A \in \mathbf{R}^{n\times n}$ be a weakly diagonally dominant matrix with entries a_{ij} that satisfies

$$|a_{ii}| \ge \sum_{j\ne i} |a_{ij}| \qquad \text{for} \qquad i = 1,\ldots,n$$

with strict inequality holding for at least one i.

(i) Suppose $Ax = 0$ for some $x \in \mathbf{R}^n$. Prove that

$$|a_{ii}x_i| \le \sum_{j\ne i} |a_{ij}||x_j| \qquad \text{for} \qquad i = 1,\ldots,n.$$

(ii) Let $\mu = \max\{\, |x_j| : j = 1,\ldots,n \,\}$ and choose i_0 so that $|x_{i_0}| = \mu$. Show

$$|a_{ii}x_i| \ge \sum_{j\ne i} |a_{ij}||x_j| \qquad \text{for} \qquad i = i_0.$$

(iii) Show that $|x_j| = \mu$ for every j such that $a_{i_0 j} \ne 0$.

(iv) Show that if every element of A is non-zero then A is invertible.

(v) Show, even if some elements of A are zero, that if the upper and lower diagonals $a_{i+1,i} \ne 0$ and $a_{i,i+1} \ne 0$ for $i = 1,\ldots,n-1$ then A is invertible.

2. It is known that if $|p(x)| \le R$ and $q(x) \le 0$ for $x \in [a,b]$ then the two-point boundary value problem

$$y'' + p(x)y' + q(x)y = f(x) \qquad \text{where} \qquad y(a) = A, \quad y(b) = B$$

has a unique solution. Let $h = (b-a)/m$ and define $x_k = a + kh$. The matrix

$$\tilde{L} = \begin{bmatrix} a_1 & c_1 & 0 & \cdots & 0 \\ b_1 & a_2 & c_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{m-3} & a_{m-2} & c_{m-2} \\ 0 & \cdots & 0 & b_{m-2} & a_{m-1} \end{bmatrix}$$

where $a_k = -2 + h^2 q(x_k)$, $b_{k-1} = 1 - hp(x_k)/2$ and $c_k = 1 + hp(x_k)/2$ comes from the finite difference approximation

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + p(x_k)\frac{y_{k+1} - y_{k-1}}{2h} + q(x_k)y_k = f(x_k).$$

(i) Show that if $hR < 2$ then $\tilde{L}$ is invertible and conclude that the finite difference approximation also has a unique solution.

3. Take $q(x) = 0,\ a = 0,\ b = 12$ and choose $p(x)$, $f(x)$, A and B according to your UNR network identification from the table

netid	$p(x)$	$f(x)$	A	B
abelizario	$-\sin 2x$	1	2	5
ablandino	$-\sin 2x$	0	1	5
arobards	$\cos 2x$	0	3	1
austinchapman	$-\cos 2x$	1	3	2
beaus	$\sin 2x$	-1	4	3
bryanwolf	$-\sin 2x$	-1	1	0
daberasturi	$\sin x$	-1	2	0
ecoats	$-\sin 2x$	0	1	5
eguzman	$\sin 2x$	0	1	3
gharper	$-\sin 2x$	-1	5	2
ipierce	$-\cos 2x$	0	4	3
isodhi	$-\cos 2x$	1	2	1
jchou	$\sin x$	0	5	0
jdardis	$-\sin x$	1	3	1
jganska	$\sin x$	0	0	6
jludwig	$-\sin 2x$	1	5	1
jmei	$-\sin 2x$	0	3	1
jmvolk	$\sin x$	0	4	3
josephlward	$\cos 2x$	1	5	2
joyd	$-\sin x$	0	4	1
kgilgen	$-\cos 2x$	0	0	4
lbrauner	$-\cos 2x$	-1	1	4
lforbes	$\sin 2x$	1	4	2
marcmiller	$-\cos 2x$	0	5	3
mchapman	$-\cos 2x$	1	4	2
michaelap	$-\cos 2x$	1	1	2
mitchellmartinez	$-\cos x$	0	5	0
mkarr	$-\sin x$	1	1	3
pdepolo	$-\sin x$	-1	1	2
pmilham	$-\cos 2x$	-1	2	3
pwhite	$-\sin 2x$	0	0	6
rjohannsen	$-\cos x$	1	5	5
ryleyh	$\cos x$	-1	2	4
scendejas	$\cos 2x$	-1	1	3
shaylam	$\cos 2x$	-1	4	0
sshores	$\sin 2x$	-1	5	0

(i) Use the finite difference method to solve the boundary value problem. Graph your solution and find an approximation of $y(6)$ good to 5 decimal digits.

(ii) Use the shooting method with RK4 to solve the boundary value problem. Graph your solution and find an approximation of $y(6)$ good to 5 decimal digits.

4. [Extra Credit and Math/CS 666] Consider the two-point boundary value problem

$$y'' = \mathcal{F}(x, y, y') \qquad \text{where} \qquad y(a) = A, \qquad y(b) = B.$$

In general, this problem may have many solutions or none. The shooting method treats this second order boundary value problem as a first-order initial value problem by defining $v = y'$ to obtain the system

$$\begin{cases} y' = v \\ v' = \mathcal{F}(x, y, v) \end{cases}$$

with initial conditions $y(a) = A$ and $v(a) = A'$ where A' is unknown. In general, solutions to such initial value problems are unique, however, there may be many choices for A' such that the resulting solution satisfies $y(b) = B$.

(i) Suppose $\mathcal{F}(x, y, v) = \sin(xyv) - y$, $a = 0$, $A = 0$, $b = 3$ and $B = 0$. Use the shooting method to determine how many solutions there are to the corresponding two-point boundary value problem. Draw a graph of each solution.