Math 466/666: Homework Assignment 1

This homework explores some properties of the Chebyshev polynomials related to optimal interpolation.

Students are encouraged to work together and consult resources outside of the required textbook for this assignment. Please cite any sources you consulted, including Wikipedia, other books, online discussion groups as well as personal communications. Be prepared to independently answer questions concerning the material on quizzes and exams.

Unless a disability makes it difficult, present all pencil-and-paper work in your own hand writing. To do this scan handwritten pages using a cell phone, document camera or flatbed scanner. Alternatively, you may write on a digital tablet with a writing stylus. If a computer was used to solve any part of a problem, include the code, input and output. Please upload your work as a single pdf file to WebCampus.

Equation (3.3–3) from the text states a theorem on the error in the approximations obtained using interpolating polynomials. When discussed in class we stated this result as

Theorem on Interpolating Polynomials: Given the distinct points x_i where $i = 1, \ldots, n$, let $p(x)$ be the unique interpolating polynomial of degree less than or equal $n-1$ such that

$$p(x_i) = f(x_i) \qquad \text{for} \qquad i = 1, \ldots, n.$$

Provided f has n derivatives, then for every t there is a corresponding ξ between $\min(t, x_1, \ldots, x_n)$ and $\max(t, x_1, \ldots, x_n)$ such that

$$f(t) = p(t) + \frac{q(t)}{n!} f^{(n)}(\xi) \qquad \text{where} \qquad q(t) = \textstyle\prod_{i=1}^{n}(t - x_i).$$

In this assignment we will consider how the choice of the points x_i affects $q(t)$ and the resulting bounds on the error in the approximation.

1. Consider the functions $g(t) = (t - c + \epsilon)(t - c - \epsilon)$ and $h(t) = (t - c)^2$. It is true or false that $g(t) < h(t)$ for all c, t and $\epsilon \neq 0$? If true explain why using mathematical reasoning, if false provide values of c, t and $\epsilon \neq 0$ such that $g(t) \geq h(t)$.

2. Consider the function

$$M(x_1, \ldots, x_n) = \max \left\{ \textstyle\prod_{i=1}^{n} |t - x_i| : t \in [-1, 1] \right\}.$$

For $n = 2$ find a choice for c_1 and c_2 such that

$$M(c_1, c_2) \leq M(x_1, x_2) \qquad \text{for all} \qquad x_1, x_2 \in \mathbf{R}.$$

In other words, find values c_1 and c_2 for x_1 and x_2 such that $M(x_1, x_2)$ is minimal.

3. Repeat the previous problem for $n = 3$ and $n = 4$. Please show all work and clearly explain your reasoning.

4. [Extra Credit and Math 666] Given arbitrary $n \in \mathbf{N}$, let the c_i be chosen such that

$$M(c_1, \ldots, c_n) \le M(x_1, \ldots, x_n) \qquad \text{for all} \qquad x_i \in \mathbf{R}.$$

Explain why the c_i must be distinct. Hint: Use your answer to the first question.

5. Use the angle addition formulas

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

and the Pythagorean theorem $\sin^2 a + \cos^2 a = 1$ to show $\cos(2a) = 2\cos^2 a - 1$.

6. Use techniques similar to those employed in the previous problem to express $\cos(3a)$ and $\cos(4a)$ as a function of $\cos a$.

7. [Extra Credit and Math 666] Given arbitrary $n \in \mathbf{N}$ with $n \ge 2$ use trigonometry to obtain the general reduction formula

$$\cos(na) = 2\cos a \cos\big((n-1)a\big) - \cos\big((n-2)a\big).$$

8. Define the Chebyshev polynomials as

$$T_n(t) = 2t\,T_{n-1}(t) - T_{n-2}(t) \qquad \text{where} \qquad T_0(t) = 1 \quad \text{and} \quad T_1(t) = t.$$

Find $T_2(t)$, $T_3(t)$ and $T_4(t)$.

9. Compare $T_n(t)$ to $\prod_{i=1}^{n}(t - c_i)$ for the values of c_i found earlier when $n = 2, 3, 4$. How are these functions related?

10. Compare $T_n(t)$ to the trigonometric identities for $\cos(na)$ when $n = 2, 3, 4$ by making the identification $t = \cos a$. How are these functions related?

11. The Chebyshev approximation theory implies that

$$c_i = \cos\left(\frac{\pi(i - \frac{1}{2})}{n}\right) \qquad \text{for} \qquad i = 1, \ldots, n$$

leads to the minimal value of M such that

$$M(c_1, \ldots, c_n) \le M(x_1, \ldots, x_n) \qquad \text{for all} \qquad x_i \in \mathbf{R}.$$

Verify these values of c_i agree with those found in earlier for $n = 2, 3, 4$.

12. Given an arbitrary interval $[a, b]$ define

$$\widetilde{M}(x_1, \ldots, x_n) = \max\left\{ \textstyle\prod_{i=1}^{n} |t - x_i| : t \in [a, b] \right\}$$

and let $\widetilde{c}_i$ be chosen such that

$$\widetilde{M}(\widetilde{c}_1, \ldots, \widetilde{c}_n) \le \widetilde{M}(x_1, \ldots, x_n) \qquad \text{for all} \qquad x_i \in \mathbf{R}.$$

Find a relationship between $\widetilde{c}_i$ and the values of c_i defined earlier.

#1. Let $g(t) = (t-c+\varepsilon)(t-c-\varepsilon)$ and $h(t) = (t-c)^2$. Then it is TRUE that $g(t) < h(t)$ for all c, t and $\varepsilon \neq 0$. This can be seen by subtracting

$$h(t) - g(t) = (t-c)^2 - (t-c+\varepsilon)(t-c-\varepsilon)$$

$$= (t-c)^2 - ((t-c)+\varepsilon)((t-c)-\varepsilon)$$

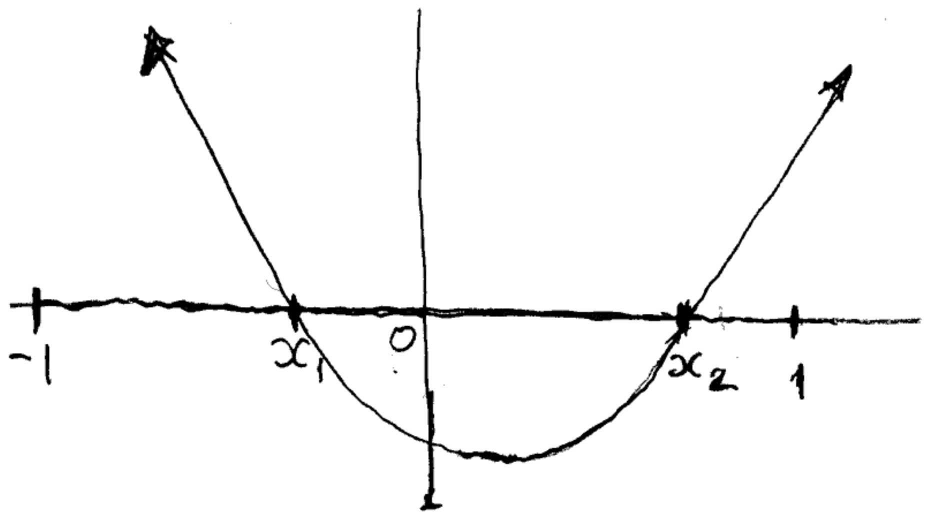
$$= \cancel{(t-c)^2} - \cancel{(t-c)^2} + \varepsilon^2 = \varepsilon^2 > 0$$

Therefore $g(t) < h(t)$.

#2. When $n=2$ we have

$$M(x_1, x_2) = \max\{ |(t-x_1)(t-x_2)| : t \in [-1,1] \}.$$

Note that the maximum of $|(t-x_1)(t-x_2)|$ occurs at either a minimum or maximum of $(t-x_1)(t-x_2)$. This is a quadratic which is concave up and look, in general as



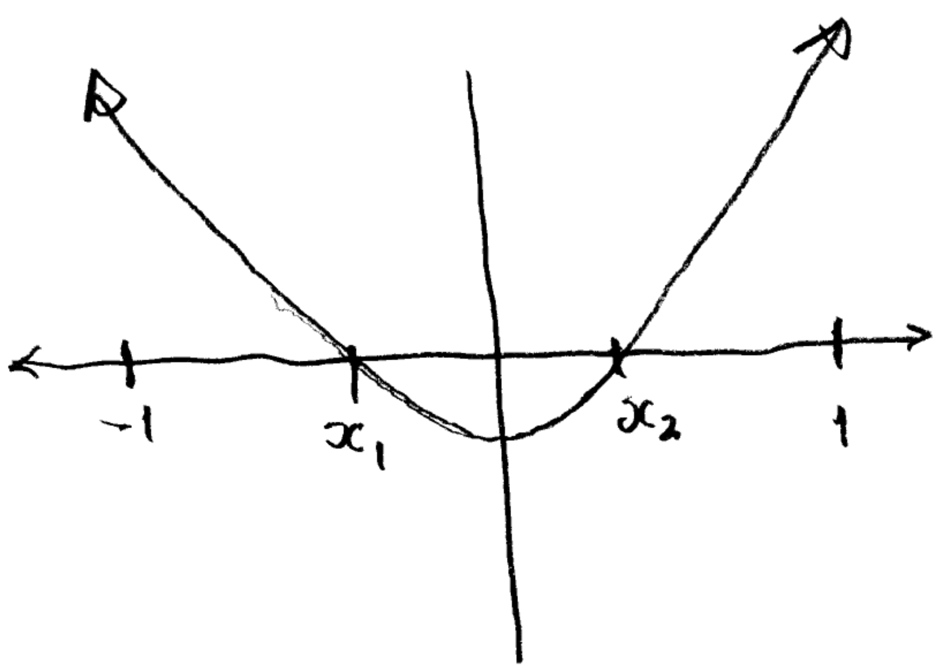
Clearly the extrema of this function occur at the endpoints of the interval and at the vertex. We now employ symmetry in order to simplify choosing the values of x_1 and x_2 that minimize $M(x_1, x_2)$.

Let $p(t) = (t-x_1)(t-x_2)$

#2 continues...

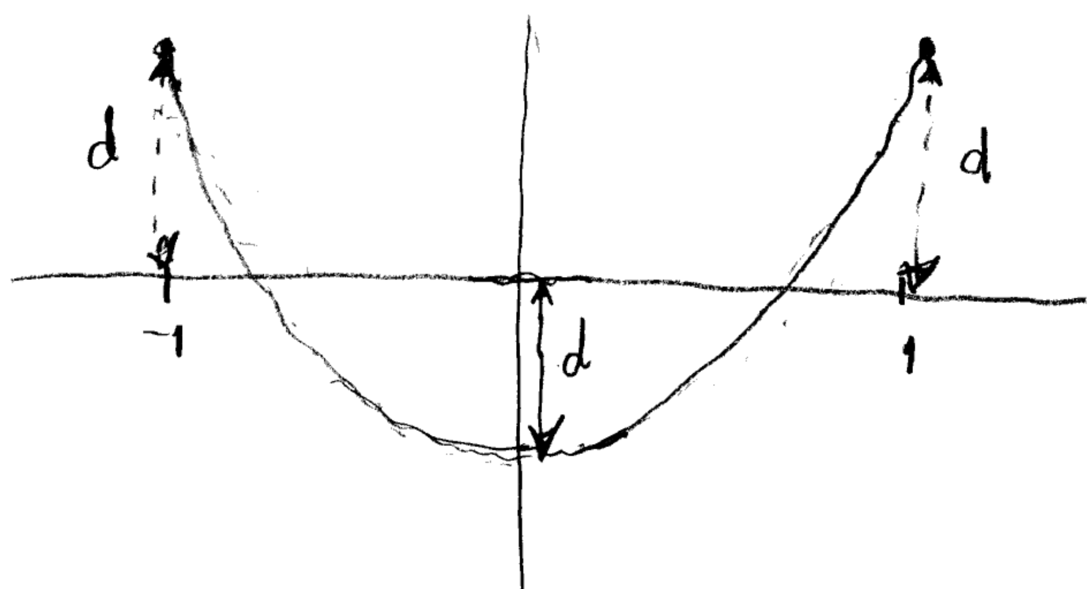
Note first that the maximum is either at $t=-1$ or $t=1$. If these values are not equal, then the larger can be reduced while the smaller increase so the are the same. This does not change the minimum but reduces the maximum. Thus, we assume that $p(-1)=p(1)$ when x_1 and x_2 have been chosen so that $M(x_1,x_2)$ is minimal.

It is easy to see this condition further implies that $x_1=-x_2$ and that the vertex occurs at $t=0$. In this case, the graph of $p(t)$ looks like



#2 continues...

We now claim that the distance from the x-axis to the maximum at $p(-1)$ or $p(1)$ must be the same as the distance to the minimum at $p(0)$. If not, then one could translate the graph either up or down and further decrease the value of $M(x_1, x_2)$. Therefore, we have



where all the distances indicated by the letter d are the same. This means, in particular, that

$$p(-1) = p(1) = -p(0).$$

#2 continues...

The symmetry arguments have led to two equations

$$p(-1) = -p(0) \quad \text{and} \quad p(1) = -p(0)$$

which are now enough to determine the optimal values of x_1 and x_2.

$$(-1-x_1)(-1-x_2) = -x_1 x_2$$
$$(1-x_1)(1-x_2) = -x_1 x_2$$

yields

$$1 + x_1 + x_2 + x_1 x_2 = -x_1 x_2$$
$$1 - x_1 - x_2 + x_1 x_2 = -x_1 x_2$$

So

$$x_1 + x_2 = -1 - 2x_1 x_2$$
$$x_1 + x_2 = 1 + 2x_1 x_2$$
$$\overline{2x_1 + 2x_2 = 0}$$

This means $x_1 = -x_2$.

#2 continues...

Substituting $x_1 = -x_2$ into one of the equations then yields

$$0 = 1 - 2x_2^2$$

So $\quad x_2 = \dfrac{1}{\pm\sqrt{2}}$ and $x_1 = \dfrac{1}{\mp\sqrt{2}}$.

For definiteness, and to preserve the ordering $x_1 < x_2$ in the graph, take

$$c_1 = -\frac{1}{\sqrt{2}} \quad \text{and} \quad c_2 = \frac{1}{\sqrt{2}}$$

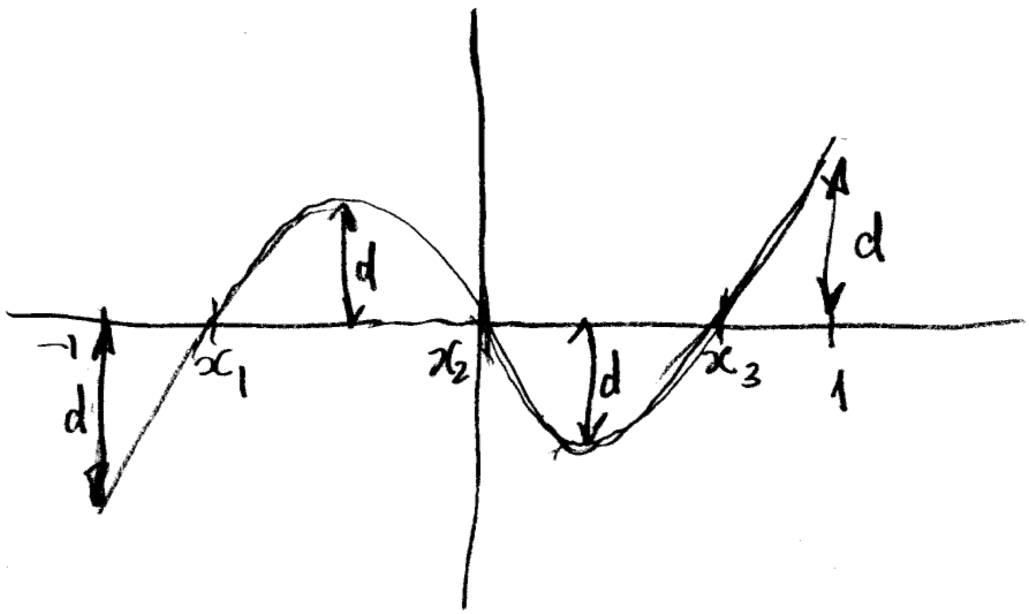
as the values for x_1 and x_2.

#3. Repeat the procedure for $n=3$ and $n=4$.

When $n=3$ similar symmetry arguments imply that the graph of

$$p(t) = (t-x_1)(t-x_2)(t-x_3)$$

looks like



In this case $x_1 = -x_3$ and $x_2 = 0$. Thus

$$p(t) = (t+x_3)t(t-x_3)$$
$$= t^3 - x_3^2 t$$

Differentiate to find the two relative extrema where $p'(t)=0$.

3. continues...

$$p'(t) = 3t^2 - x_3^2 = 0$$

Therefore $t = \pm \frac{x_3}{\sqrt{3}}$ is the location of the extrema. It follows from the fact that all the distances marked by d in the graph are equal, that

$$p(1) = -p\left(\frac{x_3}{\sqrt{3}}\right)$$

or

$$1 - x_3^2 = -\left(\left(\frac{x_3}{\sqrt{3}}\right)^3 - x_3^2\left(\frac{x_3}{\sqrt{3}}\right)\right)$$

It looks complicated, because of all the $\sqrt{3}$'s, but these can be made to disappear with the substitution

$$\alpha = \frac{x_3}{\sqrt{3}} \quad \text{so} \quad x_3^2 = 3\alpha^2.$$

Thus

$$1 - 3\alpha^2 = -\left(\alpha^3 - 3\alpha^2\alpha\right)$$

or

$$2\alpha^3 + 3\alpha^2 - 1 = 0$$

To solve this cubic, we use the rational root theorem (or a guess).

#3 continues...

Solving for α in

$$2\alpha^3 + 3\alpha^2 - 1 = 0$$

by the rational root theorem suggests we try ± 1 and $\pm \frac{1}{2}$.

Plugging in these guesses we see that $\alpha = -1$ is a solution. Thus $\alpha + 1$ is a factor and

$$\begin{array}{r|l}
 & 2\alpha^2 + \alpha - 1 \\
\hline
\alpha + 1 & 2\alpha^3 + 3\alpha^2 - 1 \\
 & \underline{2\alpha^3 + 2\alpha^2} \\
 & \alpha^2 - 1 \\
 & \underline{\alpha^2 + \alpha} \\
 & -\alpha - 1
\end{array}$$

Shows that we can solve

$$2\alpha^2 + \alpha - 1 = 0$$

to find the remaining roots.

#3 continues...

$$2\alpha^2+\alpha-1=0$$

By the quadratic formula

$$\alpha = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$$

So the remaining roots are

$$\alpha = -1 \quad \text{and} \quad \alpha = \frac{1}{2}.$$

Since $x_3 > 0$ we take $\alpha = \frac{1}{2}$ and obtain

$$x_3 = \frac{\sqrt{3}}{2}$$

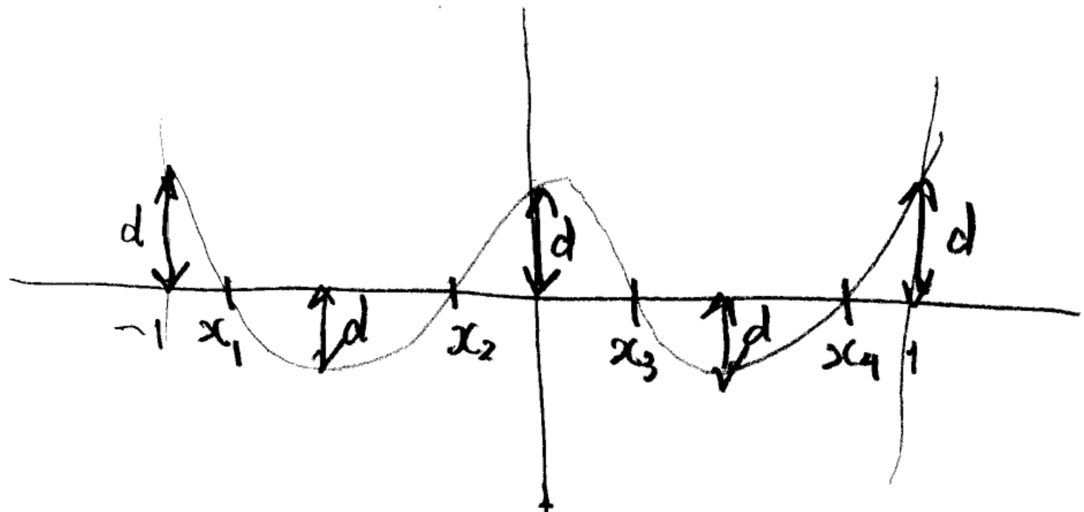
From this it follows that

$$c_1 = \frac{-\sqrt{3}}{2}, \quad c_2 = 0 \quad \text{and} \quad c_3 = \frac{\sqrt{3}}{2}$$

is the choice of x_i's for which the function $M(x_1, x_2, x_3)$ is minimized.

#3 continues...

When $n=4$ symmetry gives



Clearly the choice of x_i's that minimize $M(x_1,\dots,x_4)$ should lead to a polynomial $p(t)$ that is even. Therefore, we take

$$x_1 = -x_4 \quad \text{and} \quad x_2 = -x_3$$

Consequently

$$p(t) = (t^2 - x_3^2)(t^2 - x_4^2)$$

Since $p(0) = p(1)$ then

$$x_3^2 x_4^2 = (1-x_3^2)(1-x_4^2)$$

or $$x_3^2 + x_4^2 = 1.$$

#3 continues...

This reduces the polynomial to

$$p(t) = t^4 - t^2 + x_3^2 x_4^2$$

For simplicity, note that $d = x_3^2 x_4^2$.

Thus $$p(t) = t^4 - t^2 + d.$$

Now, differentiate to find the other extrema:

$$p'(t) = 4t^3 - 2t = 0$$

So

$$t = 0 \quad \text{or} \quad t = \pm \frac{1}{\sqrt{2}}$$

It follows that

$$-p\left(\frac{1}{\sqrt{2}}\right) = d$$

or that

$$-\frac{1}{4} + \frac{1}{2} - d = d$$

$$\frac{1}{4} = 2d \quad \text{or} \quad d = \frac{1}{8}$$

#3 continues...

Consequently, we have

$$\begin{cases} x_3^2 + x_4^2 = 1 \\ x_3^2 x_4^2 = \frac{1}{8} \end{cases}$$

Substituting, yields

$$x_3^2(1-x_3^2) = \frac{1}{8}$$

$$x_3^4 - x_3^2 + \frac{1}{8} = 0$$

which is quadratic in x_3^2. Thus

$$x_3^2 = \frac{1 \pm \sqrt{1 - 1/2}}{2} = \frac{1 \pm \sqrt{1/2}}{2}$$

Taking x_3 to be the smaller of the two options then yields that

$$c_1 = -\sqrt{\frac{1+\sqrt{1/2}}{2}}, \quad c_2 = -\sqrt{\frac{1-\sqrt{1/2}}{2}},$$

$$c_3 = \sqrt{\frac{1-\sqrt{1/2}}{2}}, \quad c_4 = \sqrt{\frac{1+\sqrt{1/2}}{2}}.$$

4. Suppose

$M(c_1, \dots, c_n) \leq M(x_1, \dots, x_n)$ for all $x_i \in \mathbb{R}$.

Explain why the c_i's must be distinct.

For contradiction, suppose they were not distinct. Then there would be at least two c_i's that were the same. Without loss of generality, suppose $c_1 = c_2 = c$.

Now consider the polynomial

$$P_\varepsilon(t) = (t-c-\varepsilon)(t-c+\varepsilon)\prod_{i=3}^{n}(t-c_i)$$

Note that $P_0(t) = \prod_{i=1}^{n}(t-c_i)$ and consequently

$$M(c_1, \dots, c_n) = \max\{|P_0(t)| : t \in [-1,1]\}$$

Let $B = \max\left\{\left|\prod_{i=3}^{n}(t-c_i)\right| : t \in [-1,1]\right\}$

and choose ε so small that

$$\varepsilon^2 B < \frac{M(c_1, \dots, c_n)}{2}$$

#4 continues ...

Note for $t \in [c-\varepsilon, c+\varepsilon]$ that

$$|(t-c-\varepsilon)(t-c+\varepsilon)|$$
$$= -(t-c-\varepsilon)(t-c+\varepsilon)$$
$$= -(t-c)^2 + \varepsilon^2 \leq \varepsilon^2$$

Therefore if $t \in [c-\varepsilon, c+\varepsilon]$, then

$$|P_\varepsilon(t)| \leq \varepsilon^2 \left| \prod_{i=3}^{n} (t-c_i) \right|$$
$$\leq \varepsilon^2 B \leq \frac{1}{2} M(c_1, \dots, c_n)$$

On the other hand, if $t \in [-1, c-\varepsilon] \cup [c+\varepsilon, 1]$ then

$$|(t-c-\varepsilon)(t-c+\varepsilon)|$$
$$= (t-c-\varepsilon)(t-c+\varepsilon)$$
$$= (t-c)^2 - \varepsilon^2 .$$

#4 continues.

In this case

$$|P_\varepsilon(t)| = \left((t-c)^2 - \varepsilon^2\right)\left|\prod_{i=1}^{3}(t-c_i)\right|$$

$$= \left(1 - \frac{\varepsilon^2}{(t-c)^2}\right)|P_0(t)|.$$

Now, since $t \in [-1, c-\varepsilon] \cup [c+\varepsilon, 1]$ and $c \in [-1,1]$ it follows that

$$(t-c)^2 \leq 2^2 = 4$$

and moreover that

$$\left(1 - \frac{\varepsilon^2}{(t-c)^2}\right) \leq 1 - \frac{\varepsilon^2}{4}$$

Consequently

$$\mathrm{Max}\{|P_\varepsilon(t)| : t \in [-1, c-\varepsilon] \cup [c+\varepsilon, 1]\}$$

$$\leq \left(1 - \frac{\varepsilon^2}{4}\right)\mathrm{Max}\{|P_0(t)| : t \in [-1, c-\varepsilon] \cup [c+\varepsilon, 1]\}$$

$$\leq \left(1 - \frac{\varepsilon^2}{4}\right) M(c_1, \dots, c_2)$$

#4 continues

Define

$$\gamma = \max\left\{\frac{1}{2}, 1 - \frac{\varepsilon^2}{4}\right\}.$$

Then $\gamma < 1$ and furthermore

$$\max\left\{|P_\varepsilon(t)| : t \in [-1,1]\right\}$$

$$\leq \gamma M(c_1, \dots, c_2) < M(c_1, \dots, c_n)$$

which contradicts $M(c_1, \dots, c_n)$ being minimal. Therefore, there can't be any repeated values among the c_i's and we conclude the are all distinct.

#5. Recall that

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

and

$$\sin^2 a + \cos^2 a = 1.$$

It follows that

$$\cos(2a) = \cos(a+a) = \cos a \cos a - \sin a \sin a$$
$$= \cos^2 a - \sin^2 a$$
$$= \cos^2 a - (1 - \cos^2 a)$$
$$= 2\cos^2 a - 1.$$

#6. Express $\cos(3a)$ and $\cos(4a)$ as functions of only $\cos a$.

$$\cos(3a) = \cos(2a + a)$$

$$= \cos 2a \cos a - \sin 2a \sin a$$

$$\sin 2a = \sin(a+a) = \sin a \cos a + \cos a \sin a$$

$$= 2 \sin a \cos a$$

$$\cos(3a) = (2\cos^2 a - 1)\cos a - 2\sin^2 a \cos a$$

$$= 2\cos^3 a - \cos a - 2(1 - \cos^2 a)\cos a$$

$$= 2\cos^3 a - 3\cos a + 2\cos^3 a$$

$$= 4\cos^3 a - 3\cos a$$

$$\cos 4a = \cos(2a + 2a) = 2\cos^2(2a) - 1$$

$$= 2(2\cos^2 a - 1)^2 - 1$$

$$= 8\cos^4 a - 8\cos^2 a + 2 - 1$$

$$= 8\cos^4 a - 8\cos^2 a + 1$$

#7 Show that

$$\cos(na) = 2\cos a \cos(n-1)a - \cos(n-2)a$$

By trigonometry

$$\cos(na) = \cos(a + (n-1)a)$$

$$= \cos a \cos(n-1)a - \sin a \sin(n-1)a$$

$$\sin(n-1)a = \sin(a + (n-2)a)$$

$$= \sin a \cos(n-2)a + \cos a \sin(n-2)a$$

Thus

$$\cos(na) = \cos a \cos(n-1)a$$

$$- \sin a \left(\sin a \cos(n-2)a + \cos a \sin(n-2)a \right)$$

$$= \cos a \cos(n-1)a - \sin^2 a \cos(n-2)a$$

$$- \sin a \cos a \sin(n-2)a$$

Now $\sin^2 a = 1 - \cos^2 a$

and

$$\sin a \sin(n-2)a = \cos a \cos(n-2)a - \cos(n-1)a$$

#7 continues...

Substituting yields

$$\cos(na) = \cos a \cos(n-1)a - (1-\cos^2 a)\cos(n-2)a$$
$$- \cos a \left(\cos a \cos(n-2)a - \cos(n-1)a\right)$$

$$= 2\cos a \cos(n-1)a - \cos(n-2)a$$

#8 Define

$$T_n(t) = 2tT_{n-1}(t) - T_{n-2}(t)$$

where $T_0(t)=1$ and $T_1(t)=t$.

Then

$$T_2(t) = 2tT_1(t) - T_0(t)$$

$$= 2t(t) - 1 = 2t^2 - 1.$$

and

$$T_3(t) = 2tT_2(t) - T_1(t)$$

$$= 2t(2t^2-1) - t$$

$$= 4t^3 - 3t.$$

and

$$T_4(t) = 2tT_3(t) - T_2(t)$$

$$= 2t(4t^3-3t) - (2t^2-1)$$

$$= 8t^4 - 8t^2 + 1.$$

#9

n	T_n	$\prod_{i=1}^{n}(t-c_i)$
2	$2t^2-1$	$\left(t+\frac{1}{\sqrt{2}}\right)\left(t-\frac{1}{\sqrt{2}}\right)$
3	$4t^3-3t$	$\left(t+\frac{\sqrt{3}}{2}\right)t\left(t-\frac{\sqrt{3}}{2}\right)$
4	$8t^4-8t^2+1$	$\left(t+\sqrt{\frac{1+\sqrt{1/2}}{2}}\right)\left(t+\sqrt{\frac{1-\sqrt{1/2}}{2}}\right)\left(t-\sqrt{\frac{1-\sqrt{1/2}}{2}}\right)\left(t-\sqrt{\frac{1+\sqrt{1/2}}{2}}\right)$

Multiplying out the polynomials in the last column obtains

$$\left(t+\frac{1}{\sqrt{2}}\right)\left(t-\frac{1}{\sqrt{2}}\right) = t^2-\frac{1}{2} = \frac{1}{2}\left(2t^2-1\right) = \frac{1}{2}T_2(t)$$

and

$$\left(t+\frac{\sqrt{3}}{2}\right)t\left(t-\frac{\sqrt{3}}{2}\right) = t\left(t^2-\frac{3}{4}\right) = \frac{1}{4}\left(4t^3-3t\right)$$

$$= \frac{1}{4}T_3(t)$$

#9 continues...

For $n=4$

$$\left(t+\sqrt{\frac{1+\sqrt{1/2}}{2}}\right)\left(t+\sqrt{\frac{1-\sqrt{1/2}}{2}}\right)\left(t-\sqrt{\frac{1-\sqrt{1/2}}{2}}\right)\left(t-\sqrt{\frac{1+\sqrt{1/2}}{2}}\right)$$

$$=\left(t^2-\frac{1+\sqrt{1/2}}{2}\right)\left(t^2-\frac{1-\sqrt{1/2}}{2}\right)$$

$$=t^4-t^2+\left(\frac{1+\sqrt{1/2}}{2}\right)\left(\frac{1-\sqrt{1/2}}{2}\right)$$

$$=t^4-t^2+\frac{1-1/2}{4}$$

$$=t^4-t^2+\frac{1}{8}$$

$$=\frac{1}{8}\left(8t^4-8t^2+1\right)=\frac{1}{8}T_4(t)$$

In Summary

$$T_n(t)=2^{n-1}\prod_{i=1}^{n}(t-c_i)$$

for $n=2,3,4$ as shown above.

#10 Recall

$$\cos(2a) = 2\cos^2 a - 1$$

$$\cos(3a) = 4\cos^3 a - 3\cos a$$

$$\cos(4a) = 8\cos^4 a - 8\cos^2 a + 1$$

Substituting $t = \cos a$ yields tat

$$\cos(2a) = T_2(\cos a)$$

$$\cos(3a) = T_3(\cos a)$$

$$\cos(4a) = T_4(\cos a)$$

as these are exactly the same coeficients in the polynomials $T_n(t)$.

#11 The Chebyshev approximation theory implies

$$C_i = \cos\left(\frac{\pi(i-1/2)}{n}\right)$$

for $i = 1, \dots, n$.

Verify these values for $n = 2, 3, 4$ with the results that minimize $M(x_1, \dots, x_n)$ earlier

From before in problem 2

$n = 2$ implies

$$C_1 = -\frac{1}{\sqrt{2}}, \quad C_2 = \frac{1}{\sqrt{2}}.$$

on the other hand

$$i=1, \quad \cos\left(\frac{\pi(1-1/2)}{2}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

and

$$i=2, \quad \cos\left(\frac{\pi(1+1/2)}{2}\right) = \cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Therefore, except for relabeling the indices, these are the same values for C_1 and C_2.

#11 continues

For $n=3$ problem 3 implies

$$C_1 = -\frac{\sqrt{3}}{2}, \quad C_2 = 0 \quad \text{and} \quad C_3 = \frac{\sqrt{3}}{2}.$$

On the other hand

$$i=1, \quad \cos\left(\frac{\pi(1-1/2)}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$i=2, \quad \cos\left(\frac{\pi(2-1/2)}{3}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$i=3, \quad \cos\left(\frac{\pi(3-1/2)}{3}\right) = \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

Again, except for relabeling the indices, these are the same values for C_1, C_2 and C_3.

#11 continues...

For $n=4$ problem 4 implies

$$c_1 = -\sqrt{\frac{1+\sqrt{1/2}}{2}}, \quad c_2 = -\sqrt{\frac{1-\sqrt{1/2}}{2}},$$

$$c_3 = \sqrt{\frac{1-\sqrt{1/2}}{2}}, \quad c_4 = \sqrt{\frac{1+\sqrt{1/2}}{2}}$$

For simplicity it seems reasonable to compare the decimal approximations

$$c_1 = -\sqrt{\frac{1+\sqrt{1/2}}{2}} \approx -0.9238795325\underline{112867}\ldots$$

$$c_2 = -\sqrt{\frac{1-\sqrt{1/2}}{2}} \approx -0.3826834323650897\ldots$$

$c_3 = -c_2$ and $c_4 = -c_1$

slightly different

with

$$i=1, \quad \cos\left(\frac{\pi(1-1/2)}{4}\right) = \cos\left(\frac{\pi}{8}\right)$$

$$\approx 0.9238795325\underline{112867}\ldots$$

$$i=2, \quad \cos\left(\frac{\pi(2-1/2)}{4}\right) = \cos\left(\frac{3\pi}{8}\right)$$

$$\approx 0.3826834323650898 4\ldots$$

almost the same as $-c_2$

#11 continues...

$i=3$, $\cos\left(\frac{\pi(3-1/2)}{4}\right) = \cos\left(\frac{5\pi}{8}\right)$

$\approx -0.38263432365089\underline{7}...$

Same as C_2...

$i=4$, $\cos\left(\frac{\pi(4-1/2)}{4}\right) = \cos\left(\frac{7\pi}{8}\right)$

$\approx -0.923879532511286\underline{7}...$

Same as C_1

Subject to slight differences in what appears to be rounding error, these are the same values for C_1, C_2, C_3 and C_4.

#12 Given an arbitrary interval $[a,b]$ define

$$\tilde{M}(x_1,\dots,x_n) = \max\left\{ \prod_{i=1}^{n} |t-x_i| : t\in[a,b] \right\}$$

and let $\tilde{c}_i$ be chosen such that

$$\tilde{M}(\tilde{c}_1,\dots,\tilde{c}_n) \leq \tilde{M}(x_1,\dots,x_n)$$

for all $x_i \in \mathbb{R}$. Find a relationship between the values of $\tilde{c}_i$ and the values c_i defined earlier.

Since rescaling the x-axis maps polynomials into polynomials (of the same degree) then it is enough to map c_i to $\tilde{c}_i$ using the same scaling relation that maps $[-1,1]$ onto $[a,b]$. Namely

$$f(t) = \alpha t + \beta$$

where

$$f(-1) = -\alpha + \beta = a$$

$$f(1) = \alpha + \beta = b$$

Thus $2\beta = a+b$ and $\beta = \frac{a+b}{2}$

$2\alpha = b-a$ and $\alpha = \frac{b-a}{2}$.

#12 continues...

The function

$$f(t) = \frac{b-a}{2} t + \frac{b+a}{2}$$

maps $f : [-1,1] \to [a,b]$ linearly.

Therefore

$$\tilde{c}_i = f(c_i) = \frac{b-a}{2} c_i + \frac{b+a}{2}$$

gives the relationship between $\tilde{c}_i$ and the values of c_i.