Math 466/666 Finite Differences of a Polynomial

Given a grid $x_j = x_0 + jh$ where $h > 0$ let $f_j = f(x_j)$ for $j \in \mathbf{Z}$. Define the shift operator E, the forward difference Δ, the backward difference ∇ and the central difference δ by

- $Ef_j = f_{j+1}$,
- $\Delta = E - I$,
- $\nabla = I - E^{-1}$,
- $\delta = E^{1/2} - E^{-1/2}$.

Note that the definition of δ involves points off the grid where, in general, we interpret non-integer powers of the shift operator as $E^\theta f_j = f(x_j + \theta h)$.

Theorem. *Suppose $f(x)$ is a polynomial of degree n then*

$$\Delta^n f_j, \qquad \nabla^n f_j \qquad \textit{and} \qquad \delta^n f_j$$

are each constants proportional to h^n.

Proof. As the treatments for Δ, ∇ and δ are similar, only the argument for Δ is presented here. The reader is invited to work out the relevant details for ∇ and δ on their own.

Since any polynomial is a linear combination of the monomial terms x^p and Δ is a linear operator, we first examine what happens when Δ is applied to x^p. Compute

$$\Delta(x_j)^p = (x_{j+1})^p - (x_j)^p = (x_j + h)^p - (x_j)^p.$$

If $p = 1$ then

$$\Delta x_j = x_j + h - x_j = h.$$

On the other hand, if $p > 1$, then use the binomial theorem to write

$$(x_j + h)^p = \sum_{k=0}^{p} \binom{p}{k} (x_j)^{p-k} h^k = (x_j)^p + \sum_{k=1}^{p} \binom{p}{k} (x_j)^{p-k} h^k$$

and obtain

$$\Delta(x_j)^p = \sum_{k=1}^{p} \binom{p}{k} (x_j)^{p-k} h^k = h \sum_{k=0}^{p-1} \binom{p}{k+1} (x_j)^{p-1-k} h^k.$$

Note in every case that

$$\Delta(x_j)^p = h \cdot (\text{a polynomial of degree } p-1 \text{ in } x_j).$$

Therefore, if $f(x)$ is a polynomial of degree n then

$$\Delta f_j = h \cdot (\text{a polynomial of degree } n-1 \text{ in } x_j).$$

Proceding by induction it follows that

$$\Delta^2 f_j = h^2 \cdot (\text{a polynomial of degree } n-2 \text{ in } x_j)$$

until finally that

$$\Delta^n f_j = h^n \cdot (\text{a polynomial of degree } 0 \text{ in } x_j).$$

Since a polynomial of degree 0 is nothing but a constant, it follows that $\Delta^n f_j$ is a constant proportional to h^n as desired. ////