Math 466/666: Final Exam Version B Solutions

This exam is open book, open web-browser and open notes. Your desktop, web camera and audio are being remotely recorded by Proctorio. The difference between Quiz 1 and Quiz 2 is that Proctorio will be recording each of you individually rather than Zoom doing the same thing as part of a video conference call.

The following instructions have not changed; however, please read them anyway:

- Do not click on the I'm finished question in WebCampus until you have shown all your written work to web camera and clearly stated how many pages you will be scanning and turning in for later grading. There is no need to show any Julia work to the web camera because that has already been captured by the screen recorder.
- Starting now until after the semester ends on December 18 do not send email, text or any other type of message to anyone about questions appearing on this exam.
- If you have a paid membership to WolframAlpha, Chegg or other fee-based web service please log out of those services. As your screen will be recorded, also log out of bank accounts, personal email and online shopping sites.
- Make sure only one monitor is enabled on your computer. If you normally use two monitors, you will have to temporarily disconnect one of them during the exam as Proctorio can only record one screen at a time.
- Before starting the exam Proctorio will verify that your video, audio and desktop are properly being recording. You will then be asked to show your student identification and your desk and work environment to the web camera.
  - It is fine to have books, notes and blank paper on your desk.
  - It's better not to have any dogs, cats or or other people in the room as the presence of multiple faces might confuse the computerized face detector.
  - If your favorite cat jumps on your desk during the exam do not panic; simply remove it and continue working.
- Work each problem using pencil and paper or using a computer and Julia as appropriate. It is not allowed to send or receive email during the exam or upload questions to any web forum or homework service.
  - It is fine to use Google and similar web search engines.
  - It it fine to use the free WolframAlpha. Do not log in to a paid account.
  - You may use any non-interactive web resources during the exam.
  - You may read Stack Overflow or the Julia website, but do not post any questions to these or any similar forums during the exam.
  - It is fine to open a browser tab to read the text book; however, I would recommend downloading a `pdf` copy ahead of time and using that if needed.
- If you find an error in the test or are confused about a question, please explain carefully in writing what is wrong and include that with your best attempt at an answer.

- When you are finished
  - Make sure your pages are consecutively numbered.
  - State how may pages you will be turning in using your microphone.
  - Show your work one page at a time to the web camera.
  - Hold each page steady for a count of 10 so the web camera can focus on it.
  - There is no need to show any Julia work to the web camera because that has already been captured by the screen recorder.
- After you have shown all your written work to the web camera, return to WebCampus, answer the I'm finished question and press submit to stop the recording.
- After you have ended Proctorio you must still upload high-resolution scans of all work to Final Upload on WebCampus along with any Julia programs and computer output used to arrive at your final answers.
  - Upload all your written work as a single `pdf` file.
  - For Julia you may upload a JupyterLab notebook as a separate `ipynb` file.
  - Do not change anything before uploading copies of your work for grading.
  - Please type a note in the comment panel for the Final Upload if you notice a mistake in your work that you want to let me know about.

Everyone must complete question 1. Then for full credit complete 4 out of the 6 questions from 2–7. Graduate students taking this course as Math 666 must complete 5 questions out of the 6 from 2–7. Question 8 is extra credit.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this exam" followed by your signature as the answer to this question.

2. Let A be the matrix

$$A = \begin{bmatrix} 4 & 9 & 2 & 1 \\ 9 & -1 & 7 & 3 \\ 7 & 6 & -8 & -8 \\ 8 & 1 & 0 & -2 \end{bmatrix}.$$

Starting with $x_0 = (8, 6, -7, -8)$ perform four iterations of the power method

$$\begin{cases} y_{k+1} = Ax_k \\ x_{k+1} = y_{k+1}/\|y_{k+1}\| \end{cases}$$

where $\|x\| = \sqrt{x^T x}$ is the vector 2-norm.

(i) What is $\|y_4\|$?

(ii) What is x_4?

The Julia code

```
using LinearAlgebra
A=[4 9 2 1; 9 -1 7 3; 7 6 -8 -8; 8 1 0 -2]
x0=[8,6,-7,-8]
x=x0
for i=1:4
    global x,y
    y=A*x
    x=y/norm(y)
end
println("(i)   ||y_4|| = ",norm(y),"\n")
println("(ii)  x_4 =\n")
display(x)
```

answers both questions and produces the output

```
(i)   ||y_4|| = 13.04034550892765

(ii)  x_4 =

4-element Array{Float64,1}:
  0.14008982462750388
  0.8458964191733639
 -0.440053892020664
  0.26680828544665103
```

3. Consider the points

$$(-1,3), \qquad (2,-1) \qquad \text{and} \qquad (4,0)$$

(i) Use the Lagrange basis functions $\ell_j(x)$ to find the interpolating polynomial $p(x)$ passing through these three points.

Since there are three points we take $x = (-1,2,4)$ and $y = (3,-1,0)$ and recall that

$$p(x) = \sum_{j=1}^{3} y_j \ell_j(x) \qquad \text{where} \qquad \ell_j(x) = \prod_{i \neq j} \frac{x - x_i}{x_j - x_i}.$$

Since $y_3 = 0$ we only need $\ell_1(x)$ and $\ell_2(x)$. Since

$$\ell_1(x) = \frac{(x-2)(x-4)}{(-1-2)(-1-4)} = \frac{1}{15}\big(x^2 - 6x + 8\big)$$

$$\ell_2(x) = \frac{(x+1)(x-4)}{(2+1)(2-4)} = \frac{-1}{6}\big(x^2 - 3x - 4\big)$$

we obtain that

$$p(x) = \frac{1}{5}\big(x^2 - 6x + 8\big) + \frac{1}{6}\big(x^2 - 3x - 4\big) = \frac{11}{30}x^2 - \frac{51}{30}x + \frac{14}{15}.$$

(ii) What is the degree of $p(x)$?

The degree of $p(x)$ is clearly 2.

(iii) What is $p(0.8)$?

To check and evaluate $p(0.8)$ use the Julia code

```
xs=[-1,2,4]
ys=[3,-1,0]
p(x)=((11*x-51)*x+28)/30
println("Checking the polynomial is correct:\n")
for i=1:3
    println("p(",xs[i],")=",p(xs[i])," and should be ",ys[i])
end
println("\n(iii)  p(0.8)=",p(0.8))
```

to obtain

```
Checking the polynomial is correct:

p(-1)=3.0 and should be 3
p(2)=-1.0 and should be -1
p(4)=0.0 and should be 0

(iii)  p(0.8)=-0.19200000000000017
```

4. Suppose the matrix A has a spectrum $\sigma(A)$ consisting of the eigenvalues

$$\sigma(A) = \{-10 + 2i, -10 - 2i, 1, 16\}.$$

Let $B = (A - 3I)^{-1}$.

(i) Use the spectral mapping theorem to find $\sigma(B)$ the eigenvalues of B.

Note that $B = f(A)$ where $f(x) = 1/(x - 3)$. Therefore, by the spectral mapping theorem

$$\begin{aligned}\sigma(B) &= f\big(\sigma(A)\big) = \big\{f(-10+2i), f(-10-2i), f(1), f(16)\big\}\\ &= \Big\{\frac{1}{-13+2i}, \frac{1}{-13-2i}, \frac{1}{-2}, \frac{1}{13}\Big\} = \Big\{-\frac{13+2i}{173}, -\frac{13-2i}{173}, -\frac{1}{2}, \frac{1}{13}\Big\}.\end{aligned}$$

(ii) Does $\sigma(B)$ contain a unique eigenvalue of maximal magnitude or not? List all eigenvalues of maximal magnitude.

Since

$$\Big| -\frac{13+2i}{173}\Big| = \Big| -\frac{13-2i}{173}\Big| = \frac{1}{\sqrt{173}} \approx 0.07602859212697055$$

$$\left| -\frac{1}{2} \right| = 0.5 \qquad \text{and} \qquad \left| \frac{1}{13} \right| \approx 0.07692307692307693$$

then $\sigma(B)$ contains a unique eigenvalue $-1/2$ of maximum value.

5. Consider the function $f(x) = \cos(2x) - x^2$. Newton's method for solving $f(x) = 0$ may be expressed as the one-step iteration

$$x_{k+1} = g(x_k) \qquad \text{where} \qquad x_0 = \text{an initial approximation.}$$

(i) Explicitly write out what $g(x)$ is in this case.

By definition

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\cos(2x) - x^2}{-2\sin(2x) - 2x} = x + \frac{1}{2} \cdot \frac{\cos(2x) - x^2}{\sin(2x) + x}.$$

(ii) Perform four iterations of Newtons method starting with $x_0 = 1$. What is x_4?

The Julia code

```
g(x)=x+(cos(2*x)-x^2)/(sin(2*x)+x)/2
x0=1
x=x0
for i=1:4
    global x
    x=g(x)
end
println("(ii)  x_4=",x)
```

produces the output

```
(ii)  x_4=0.600769149670293
```

6. Let the matrix A and vector b be defined as

$$A = \begin{bmatrix} 8 & 0 & 6 & -1 \\ 0 & 10 & -2 & 1 \\ 6 & -2 & 7 & -1 \\ -1 & 1 & -1 & 8 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(i) Is A strictly diagonally dominant? If not explain why, if so provide a verification.

For A to be strictly diagonally dominant we need

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \qquad \text{for every} \qquad i = 1, \ldots, 4.$$

We verify this line by checking

$$\begin{aligned}
|a_{11}| &= 8 &\quad \text{compared with} \quad & \sum_{j\neq 1} |a_{1j}| = 0+6+1 = 7\\
|a_{22}| &= 10 &\quad \text{compared with} \quad & \sum_{j\neq 2} |a_{2j}| = 0+2+1 = 3\\
|a_{33}| &= 7 &\quad \text{compared with} \quad & \sum_{j\neq 3} |a_{3j}| = 6+2+1 = 9\\
|a_{44}| &= 8 &\quad \text{compared with} \quad & \sum_{j\neq 4} |a_{4j}| = 1+1+1 = 3.
\end{aligned}$$

Since $7 \le 9$ the condition for strict diagonal dominance fails when $i = 3$.

(ii) Starting with $x_0 = (1, 0, 0, 0)$ perform four Gauss–Seidel iterations

$$x_{k+1} = Bx_k + c \qquad \text{where} \qquad B = -L^{-1}(A - L), \qquad c = L^{-1}b$$

and $L =$ `LowerTriangular`(A) is the lower-triangular part of the matrix A along with the diagonal. What is x_4?

(iii) Let $x = x_t$ be the solution to $Ax = b$ and use the condition number $\kappa(A) \approx 11.77$ along with the inequality

$$\frac{\|e\|}{\|x_t\|} \le \kappa(A)\frac{\|r\|}{\|b\|}$$

to bound the relative error in x_4.

The Julia code

```
using LinearAlgebra
A=[8 0 6 -1; 0 10 -2 1; 6 -2 7 -1; -1 1 -1 8]
b=[1,1,1,1]
L=LowerTriangular(A)
U=A-L
x0=[1,0,0,0]
x=x0
for i=1:4
    global x
    x=L\(b-U*x)
end
println("(ii)  x_4=\n")
display(x)

kappa=11.77
r=A*x-b
println("\n\n(iii)  The relative error in x_4 is bounded by\n")
display(kappa*norm(r)/norm(b))
```

answers parts (ii) and (iii) with the output

```
(ii)  x_4=

4-element Array{Float64,1}:
 0.03525059879625503
 0.11481833416908198
 0.164979381119912
 0.13567645571838563

(iii)  The relative error in x_4 is bounded by

0.8462521106952774
```

7. Let A be the matrix

$$A = \begin{bmatrix} -1 & -1 & 5 & 9 \\ 7 & 0 & 2 & 9 \\ 3 & -6 & 1 & -1 \\ -6 & -9 & 6 & -5 \end{bmatrix}$$

and consider the Hausholder reflectors $P = I - 2vv^T$ where $v^T v = 1$.

(i) Find a Hausholder reflector P such that PA is of the form

$$PA = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

where entries with $*$ represent a number that may or may not be zero.

(ii) Verify that $P^T P = I$ and $PP^T = I$.

The Julia code

```
using LinearAlgebra
A=[-1 -1 5 9; 7 0 2 9; 3 -6 1 -1; -6 -9 6 -5]
a=A[:,1]
c=-sign(a[1])*norm(a)
I4=diagm(ones(4))
e1=I4[:,1]
v=a-c*e1
v=v/norm(v)
println("(i)  Since P=I-2vv' where v=\n")
display(v)
P=I4-2*v*v'
println("\n\nthen P=\n")
display(P)
println("\n\nChecking PA=\n")
```

```
display(P*A)
println("\n\n(ii) Now compute P'P=\n")
display(P'*P)
println("\n\nand PP'=\n")
display(P*P')
```

answers part (i) and (ii) as

```
(i)  Since P=I-2vv' where v=

4-element Array{Float64,1}:
 -0.7424950623433516
  0.4836293753880197
  0.2072697323091513
 -0.4145394646183026

then P=

4×4 Array{Float64,2}:
 -0.102598   0.718185   0.307794  -0.615587
  0.718185   0.532205  -0.200483   0.400967
  0.307794  -0.200483   0.914079   0.171843
 -0.615587   0.400967   0.171843   0.656314

Checking PA=

4×4 Array{Float64,2}:
  9.74679       3.79612  -2.46235   8.31042
  0.0          -3.12399   6.86065   9.44916
  2.22045e-16  -7.33885   3.08314  -0.807503
 -8.88178e-16  -6.3223    1.83373  -5.38499

(ii) Now compute P'P=

4×4 Array{Float64,2}:
  1.0           5.51404e-17   6.75565e-17  -1.69285e-16
  5.51404e-17   1.0          -7.43234e-17   1.70905e-16
  6.75565e-17  -7.43234e-17   1.0           1.06698e-17
 -1.69285e-16   1.70905e-16   1.06698e-17   1.0

and PP'=

4×4 Array{Float64,2}:
  1.0           5.51404e-17   6.75565e-17  -1.69285e-16
  5.51404e-17   1.0          -7.43234e-17   1.70905e-16
```

```
 6.75565e-17  -7.43234e-17   1.0          1.06698e-17
-1.69285e-16   1.70905e-16   1.06698e-17   1.0
```

Note that PA has reduced the first column so that it has zeros below the first entry. Moreover, the values of P^TP and PP^T are both equal to the identity subject to the expected rounding errors of about 15 digits of precision.

8. [Extra Credit] Let $A \in \mathbf{R}^{n\times n}$ be a symmetric matrix with $A = A^T$ and suppose ξ is a unit eigenvector of A with corresponding eigenvalue λ such that $A\xi = \lambda\xi$. Consider the set $S = \{ v \in \mathbf{R}^n : \xi^T v = 0 \}$.

(i) Given any $v \in S$ prove that $Av \in S$.

Since $v \in S$ then $\xi^T v = 0$. Now, since $A^T = A$ and ξ is an eigenvector, then

$$\xi^T Av = (A^T\xi)^T v = (A\xi)^T v = (\lambda\xi)^T v = \lambda\xi^T v = 0.$$

Therefore $Av \in S$.

(ii) Given any $v, w \in S$ and $\alpha \in \mathbf{R}$ prove that $v + w \in S$ and $\alpha v \in S$.

For the first use linearity of the dot product to obtain

$$\xi^T(v + w) = \xi^T v + \xi^T w = 0 + 0 = 0$$

which shows $v + w \in S$. Similarly $\xi^T(\alpha v) = \alpha\xi^T v = 0$ shows $\alpha v \in S$.

(iii) Given any $x \in \mathbf{R}^n$ find $v \in S$ and $y \in \mathbf{R}^n$ such that $x = v + y$ and $y^T v = 0$.

Let y be the orthogonal projection of x onto ξ. Since ξ is a unit vector then $y = (\xi^T x)\xi$. Now define $v = x - y$.

Clearly $x = v + y$. What needs to be shown is that $v \in S$. To see this compute

$$\xi^T v = \xi^T(x - y) = \xi^T\big(x - (\xi^T x)\xi\big) = \xi^T x - (\xi^T x)(\xi^T\xi) = \xi^T x - \xi^T x = 0.$$

Therefore, $v \in S$ and we have expressed x as the desired sum.

(iv) The above implies S is a subspace of $\mathbf{R}^n$ that is invariant with respect to A. Let $\{v_1, \ldots, v_{n-1}\}$ be an orthonormal basis for S and define $P \in \mathbf{R}^{n\times n}$ to be the matrix with columns given by the vectors ξ and v_j as

$$P = \begin{bmatrix} \xi & v_1 & \cdots & v_{n-1} \end{bmatrix}.$$

Show that $P^{-1}AP$ has the block diagonal form

$$P^{-1}AP = \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix}$$

where $B \in \mathbf{R}^{(n-1)\times(n-1)}$.

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First note that

$$P^T P = \begin{bmatrix} \xi^T \\ v_1^T \\ \vdots \\ v_2^T \end{bmatrix} \begin{bmatrix} \xi & v_1 & \cdots & v_{n-1} \end{bmatrix} = \begin{bmatrix} \xi^T\xi & \xi^T v_1 & \cdots & \xi^T v_{n-1} \\ v_1^T\xi & v_1^T v_1 & \cdots & v_1^T v_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n-1}^T\xi & v_{n-1}^T v_1 & \cdots & v_{n-1}^T v_{n-1} \end{bmatrix}$$

along with the orthogonality of ξ to any vector in $v_i \in S$ and the fact that the v_i's form an orthonormal basis yields that $P^T P = I$. In particular $P^{-1} = P^T$.

Calculating obtains

$$AP = A\begin{bmatrix} \xi & v_1 & \cdots & v_{n-1} \end{bmatrix} = \begin{bmatrix} \lambda\xi & Av_1 & \cdots & Av_{n-1} \end{bmatrix}$$

and therefore that

$$P^{-1}AP = \begin{bmatrix} \xi^T \\ v_1^T \\ \vdots \\ v_2^T \end{bmatrix} \begin{bmatrix} \lambda\xi & Av_1 & \cdots & Av_{n-1} \end{bmatrix} = \begin{bmatrix} \lambda\xi^T\xi & \xi^T Av_1 & \cdots & \xi^T Av_{n-1} \\ \lambda v_1^T\xi & v_1^T Av_1 & \cdots & v_1^T Av_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda v_{n-1}^T\xi & v_{n-1}^T Av_1 & \cdots & v_{n-1}^T Av_{n-1} \end{bmatrix}$$

As $v^T\xi = \xi^T v = 0$ for $v \in S$ and since we've already shown $\xi^T Av = 0$, it follows that

$$P^{-1}AP = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & v_1^T Av_1 & \cdots & v_1^T Av_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & v_{n-1}^T Av_1 & \cdots & v_{n-1}^T Av_{n-1} \end{bmatrix}.$$

Now, taking

$$B = \begin{bmatrix} v_1^T Av_1 & \cdots & v_1^T Av_{n-1} \\ \vdots & \ddots & \vdots \\ v_{n-1}^T Av_{n-1} & \cdots & v_{n-1}^T Av_{n-1} \end{bmatrix} \in \mathbf{R}^{(n-1)\times(n-1)}$$

shows $P^{-1}AP$ has the desired block structure.

(v) Is $B = B^T$ true or false? If true explain why; if false provide a counter example.

It is true that $B = B^T$. This follows directly from the symmetry of A since

$$B_{ij} = v_i^T Av_j = (Av_j)^T v_i = v_j^T A^T v_i = v_j^T Av_i = B_{ji}.$$