Math 466/666: Homework Assignment 2

This homework uses the vector and matrix 2-norms to explore some properties of matrices, eigenvalues and eigenvectors. Note that you will need to use Julia or similar software for one of these problems.

Students are encouraged to work together and consult resources outside of the required textbook for this assignment. Please cite any sources you consulted, including Wikipedia, other books, online discussion groups as well as personal communications. Be prepared to independently answer questions concerning the material on quizzes and exams.

Unless a disability makes it difficult, present all pencil-and-paper work in your own hand writing. To do this scan handwritten pages using a cell phone, document camera or flatbed scanner. Alternatively, you may write on a digital tablet with a writing stylus. If a computer was used to solve any part of a problem, include the code, input and output. Please upload your work as a single pdf file to WebCampus.

Given a matrix $A \in \mathbf{R}^{m\times n}$ recall that the matrix 2-norm is defined as

$$\|A\|_2 = \max\left\{\, \|Ax\|_2 : \|x\|_2 = 1 \,\right\}$$

where the vector 2-norms $\|Ax\|_2$ and $\|x\|_2$ are given by

$$\|Ax\|_2 = \sqrt{(Ax)^T Ax} \qquad \text{and} \qquad \|x\|_2 = \sqrt{x^T x} \qquad \text{for} \qquad x \in \mathbf{R}^n.$$

For convenience of notation we will, henceforth, drop the subscripts and write

$$\|A\| = \|A\|_2, \qquad \|Ax\| = \|Ax\|_2 \qquad \text{and} \qquad \|x\| = \|x\|_2.$$

1. Show that $\|I\| \ge 1$ where I is the identity operator.

2. Let $A \in \mathbf{R}^{n\times n}$ be invertible. Show that $\|A\|\|A^{-1}\| \ge 1$.

3. Define

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{bmatrix}.$$

Use the definition of the matrix 2-norm to compute $\|A\|$ and $\|A^T\|$.

4. Let $A \in \mathbf{R}^{7\times 4}$ be a matrix with entries chosen randomly (and independently) from the uniform distribution on the interval $[-1, 1]$. Use any numerical means you prefer to compute $\|A\|$ and $\|A^T\|$ for three different samples of the random matrices just described. For example, in Julia you could enter the commands

```
using LinearAlgebra
A=2*rand(7,4).-1;
opnorm(A)
opnorm(A')
```

three times. What relationship between $\|A\|$ and $\|A^T\|$ do you notice? Please include all program listings and computer output as part of your answer for this question.

5. Given $A \in \mathbf{R}^{m\times n}$ let $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ be unit vectors such that

$$\|Ax\| = \|A\| \qquad \text{and} \qquad \|A^T y\| = \|A^T\|.$$

Thus x and y have been chosen to be vectors for which the maximum in the definition of the matrix 2-norm is attained.

(i) Use the Cauchy–Schwarz inequality to prove that

$$\|Ax\|^2 \le \|A^T Ax\| \qquad \text{and} \qquad \|A^T y\|^2 \le \|AA^T y\|.$$

(ii) By repeated applications of the definition of the norm it follows that

$$\|AA^T y\| \le \|A\|\|A^T\| \qquad \text{and} \qquad \|A^T Ax\| \le \|A^T\|\|A\|.$$

Explain why $\|AA^T y\| = \|A^T Ax\|$ and finally why $\|A\| = \|A^T\|$.

6. Let $A \in \mathbf{R}^{m\times n}$ set $B = A^T A$ and define

$$\lambda = \max\left\{ x^T Bx : \|x\| = 1 \right\}.$$

(i) Explain why $\lambda \ge 0$ and show that $\|A\| = \|A^T\| = \sqrt{\lambda}$.

(ii) Choose $\xi \in \mathbf{R}^n$ to be a unit vector such that $\xi^T B\xi = \lambda$ and show that $\|B\xi\| \le \lambda$.

(iii) Expand the inner product $(B\xi - \lambda\xi)^T(B\xi - \lambda\xi)$ and show that $\|B\xi - \lambda\xi\| = 0$.

(iv) Is it true or false that ξ must be an eigenvector of B with λ as an eigenvalue? If true explain why; if false provide a counter example.

7. [Extra Credit and for Math 666] Let $B \in \mathbf{R}^{n\times n}$ be any symmetric matrix with $B^T = B$ and choose $\xi \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$ such that $\lambda = \max\{ x^T Bx : \|x\| = 1 \}$ and $\xi^T B\xi = \lambda$.

(i) Find an example of a matrix B such that $\lambda < 0$.

(ii) Show for any $v \in \mathbf{R}^n$ that $v^T Bv \le \lambda v^T v$.

(iii) Set $v = \xi + \epsilon w$ where $w \in \mathbf{R}^n$ and $\epsilon > 0$ and simplify to prove that

$$2w^T(B\xi - \lambda\xi) \le \epsilon(\lambda w^T w - w^T Bw).$$

(iv) Now set $w = B\xi - \lambda\xi$. Is it true or false that ξ must be an eigenvector of B with λ as an eigenvalue? If true explain why; if false provide a counter example.

#1. Show that $\|I\| \geq 1$ where I is the identity operator.

By definition

$$\|I\| = \max\{\|Ix\| : \|x\| = 1\}$$

$$= \max\{\|x\| : \|x\| = \} = 1.$$

So, in fact, $\|I\| = 1$.

#2. Let $A \in \mathbb{R}^{n\times n}$ be invertible. Show that $\|A\| \, \|A^{-1}\| \geq 1$.

Let $x \in \mathbb{R}^n$ be a unit vector. Since $x \neq 0$ then $A^{-1}x \neq 0$, for otherwise $A^{-1}x = 0$ would imply $x = AA^{-1}x = A0 = 0$ which is impossible.

Define $y = \dfrac{A^{-1}x}{\|A^{-1}x\|}$. Then $y \in \mathbb{R}^n$ is also a unit vector. By definition of matrix norm

$$\|Ay\| \leq \|A\| \quad \text{and} \quad \|A^{-1}x\| \leq \|A^{-1}\|.$$

#2 continues...

Now,

$$\frac{1}{\|A^{-1}x\|} = \frac{\|x\|}{\|A^{-1}x\|} = \frac{\|AA^{-1}x\|}{\|A^{-1}x\|} = \|Ay\| \leq \|A\|$$

implies

$$1 \leq \|A\| \, \|A^{-1}x\| \leq \|A\| \, \|A^{-1}\|$$

which was to be shown.

#3 Define

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{bmatrix}$$

Use the definition of the matrix 2-norm to compute $\|A\|$ and $\|A^T\|$.

Note that for $x \in \mathbb{R}^5$ such that $\|x\| = 1$ we have

$$Ax = \begin{bmatrix} 2x_1 \\ 3x_2 \\ 4x_3 \end{bmatrix} \in \mathbb{R}^3.$$

Therefore

$$\|Ax\|^2 = 4x_1^2 + 9x_2^2 + \overset{\text{largest}}{16}x_3^2$$

$$\leq 16(x_1^2 + x_2^2 + x_3^2)$$

$$\leq 16(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)$$

$$= 16\|x\|^2 = 16.$$

From this it follows that

#3 continues...

$$\|A\| = \max\{\|Ax\| : \|x\| = 1\}$$
$$\leq \sqrt{16} = 4$$

On the other hand, taking

$$x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^5$$

yields a unit vector $\|x\| = 1$ such that

$$Ax = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}.$$

In particular, for this particular unit vector $\|Ax\| = \left\|\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}\right\| = 4$.

Consequently $\|A\| \geq 4$.

It follows that $\|A\| = 4$.

#3 continues...

Now consider $A^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

If $y \in \mathbb{R}^3$ is a unit vector $\|y\| = 1$ we have

$$A^T y = \begin{bmatrix} 2y_1 \\ 3y_2 \\ 4y_3 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^5$$

Therefore

$$\|A^T y\|^2 = 4y_1^2 + 9y_2^2 + 16y_3^2 + 0^2 + 0^2$$
$$\leq 16(y_1^2 + y_2^2 + y_3^2) = 16\|y\|^2 = 16$$

Consequently $\|A^T\| \leq 4$. On the other hand taking $y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ yields that

$\|A^T y\| = 4$ so $\|A^T\| \geq 4$.

It again follows that $\|A^T\| = 4$.

466hw2p4

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Homework 2 Problem 4

4. Let $A \in \mathbf{R}^{7\times 4}$ be a matrix with entries chosen randomly (and independently) from the uniform distribution on the interval $[-1,1]$. Use any numerical means you prefer to compute $\|A\|$ and $\|A^T\|$ for three different samples of the random matrices just described.

[1]:
```
using LinearAlgebra
```

[7]:
```
for i=1:3
    A=2*rand(7,4).-1
    println("Random Matrix ",i)
    println("  |A| = ",opnorm(A))
    println("|A^T| = ",opnorm(A'))
    println("")
end
```

```
Random Matrix 1
  |A| = 2.013200448215746
|A^T| = 2.0132004482157457

Random Matrix 2
  |A| = 2.3881749141483133
|A^T| = 2.388174914148313

Random Matrix 3
  |A| = 2.1603780237379278
|A^T| = 2.1603780237379273
```

Note, up to rounding errors, that $\|A\| = \|A^T\|$.

[ ]:
```

```

\# 5 Given $A \in \mathbb{R}^{m \times n}$ let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ be unit vectors such that

$$\|Ax\| = \|A\| \quad \text{and} \quad \|A^T y\| = \|A^T\|.$$

(i) Use the Cauchy-Schwarz inequality to prove

$$\|Ax\|^2 \leq \|A^T A x\| \quad \text{and} \quad \|A^T y\|^2 \leq \|A A^T y\|$$

By definition of the vector norm

$$\|Ax\|^2 = (Ax)^T (Ax) = x^T A^T A x$$

$$= x^T (A^T A x).$$

Now, Cauchy-Schwarz gives

$$|x^T (A^T A x)| \leq \|x\| \, \|A^T A x\|$$

Consequently, since $\|x\| = 1$, then

$$\|Ax\|^2 \leq \|A^T A x\|.$$

#5 (i) continues...

Similarly

$$\|A^T y\|^2 = (A^T y)^T A^T y = y^T A^{TT} A^T y$$
$$= y^T (A A^T y) \leq \|y\| \|A A^T y\| = \|A A^T y\|.$$

(ii) By repeated applications of the definitions of the norm it follows

$$\|A A^T y\| \leq \|A\| \|A^T\| \text{ and } \|A^T A x\| \leq \|A^T\| \|A\|.$$

Explain why

$$\|A A^T y\| = \|A^T A x\|$$

and finally why

$$\|A\| = \|A^T\|.$$

#5 (ii) Continues...

Since $\|AA^Ty\| \le \|A\|\|A^T\|$ then

$$\|AA^Ty\|^{\cancel{2}} \le \|A\|^2 \|A^T\|^2$$

$$= \|Ax\|^2 \|A^Ty\|^2$$

$$\le \|A^TAx\| \cancel{\|AA^Ty\|}.$$

It follows that

$$\|AA^Ty\| \le \|A^TAx\|$$

Similarly

$$\|A^TAx\|^{\cancel{2}} \le \|A^T\|^2 \|A\|^2 =$$

$$= \|A^Ty\|^2 \|Ax\|^2 \le \|AA^Ty\| \cancel{\|A^TAx\|}$$

implies $\|A^TAx\| \le \|AA^Ty\|$.

Putting these two inequalities together shows that $\|A^TAx\| = \|AA^Ty\|$.

#5 (ii) For the second equality note that

$$\|A\|^2 = \|Ax\|^2 \le \|A^T A x\| \le \|A^T\| \, \|A\|$$

so $\|A\| \le \|A^T\|$ and similarly

$$\|A^T\|^2 = \|A^T y\|^2 \le \|A A^T y\| \le \|A\| \, \|A^T\|$$

so $\|A^T\| \le \|A\|$

Again putting these two inequalities together shows that $\|A\| = \|A^T\|$.

#6 Let $A \in \mathbb{R}^{m\times n}$ set $B = A^TA$ and define
$$\lambda = \max\{x^TBx : \|x\| = 1\}$$

(i) Explain why $\lambda \geq 0$ and show that
$$\|A\| = \|A^T\| = \sqrt{\lambda}.$$

Since
$$x^TBx = x^TA^TAx = (Ax)^TAx$$
$$= \|Ax\|^2 \geq 0$$
it is clear why $\lambda \geq 0$. To see the equality note that
$$\|A\| = \max\{\|Ax\| : \|x\| = 1\}$$
$$= \sqrt{\max\{\|Ax\|^2 : \|x\| = 1\}}$$
$$= \sqrt{\max\{x^TBx : \|x\| = 1\}} = \sqrt{\lambda}.$$

Since $\|A\| = \|A^T\|$, we also obtain $\|A^T\| = \sqrt{\lambda}$.

#6(ii) Choose $\xi \in \mathbb{R}^n$ to be a unit vector such that $\xi^T B \xi = \lambda$ and show that $\|B\xi\| \leq \lambda$.

$$
\begin{aligned}
\|B\xi\|^2 &= (B\xi)^T B\xi = \xi^T B^T B \xi \\
&= \xi^T (B^T B \xi) \leq \cancelto{1}{\|\xi\|} \, \|B^T B \xi\| \\
&\leq \|(A^T A)^T A^T A \xi\| = \|A^T A A^T A \xi\| \\
&\leq \|A^T\| \|A\| \|A^T\| \|A\| \\
&= \sqrt{\lambda}\sqrt{\lambda}\sqrt{\lambda}\sqrt{\lambda} = \lambda^2 .
\end{aligned}
$$

Therefore, taking square roots yields

$$\|B\xi\| \leq \lambda.$$

#6 (iii) Expand the inner product $(B\xi-\lambda\xi)^T(B\xi-\lambda\xi)$ and show that $\|B\xi-\lambda\xi\|=0$.

Expanding obtains

$$(B\xi-\lambda\xi)^T(B\xi-\lambda\xi)$$

$$=(B\xi)^T B\xi-(B\xi)^T\lambda\xi-(\lambda\xi)^T B\xi+(\lambda\xi)^T(\lambda\xi)$$

$$=\|B\xi\|^2-2\lambda\,\xi^T B\xi+\lambda^2\|\xi\|^2$$

$$\leq \lambda^2-2\lambda\cdot\lambda+\lambda^2=0.$$

Therefore

$$\|B\xi-\lambda\xi\|=0.$$

6(iv) Is it true or false that ξ must be an eigenvector of B with λ as an eigenvalue?

Since $\|B\xi - \lambda\xi\| = 0$, then it follows that

$$B\xi = \lambda\xi,$$

as this is the eigenvector-eigenvalue equation for B, the only thing left to check is that $\xi \neq 0$.

However, since $\|\xi\| = 1$, then it is clear $\xi \neq 0$.

Consequently it's true that ξ is an eigenvector of B with λ as the corresponding eigenvalue.

#7 Let $B \in \mathbb{R}^{n\times n}$ be any symmetric matrix with $B^T = B$ and choose $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that $\lambda = \max\{x^T B x : \|x\| = 1\}$ and $\xi^T B \xi = \lambda$.

(i) Find an example of a matrix B such that $\lambda < 0$.

Let $B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = -2I$.

Then $B^T = B$ and

$$x^T B x = x^T(-2Ix) = -2x^Tx$$

$$= -2\|x\|^2 = -2$$

for all unit vectors $x \in \mathbb{R}^2$. It follows that $\lambda = -2 < 0$.

(ii) Show for any $v \in \mathbb{R}^n$ that

$$v^T B v \leq \lambda v^T v.$$

#7(ii) continues...

If $v=0$ then

$$v^T B v = 0^T B 0 = 0$$

and $\lambda v^T v = \lambda 0^T 0 = 0$

So the inequality $v^T B v \leq \lambda v^T v$ holds in a trivial way.

If $v \neq 0$ then $x = \frac{v}{\|v\|}$ is a unit vector and by definition of λ we have that $x^T B x \leq \lambda$.

Substituting in terms of v yields

$$x^T B x = \left(\frac{v}{\|v\|}\right)^T B \left(\frac{v}{\|v\|}\right) = \frac{1}{\|v\|^2} v^T B v$$

Therefore

$$v^T B v = \|v\|^2 x^T B x \leq \lambda \|v\|^2 = \lambda v^T v.$$

as desired.

#7 iii) Set $v = \xi + \varepsilon w$ where $w \in \mathbb{R}^n$ and $\varepsilon > 0$ and simplify to prove that

$$2w^T(B\xi - \lambda\xi) \leq \varepsilon(\lambda w^T w - w^T B w).$$

Substituting into $v^T B v \leq \lambda v^T v$ we obtain

$$v^T B v = (\xi + \varepsilon w)^T B (\xi + \varepsilon w)$$

$$= \xi^T B \xi + \xi^T B \varepsilon w + (\varepsilon w)^T B \xi + (\varepsilon w)^T B \varepsilon w$$

$$= \lambda + \varepsilon (B^T \xi)^T w + \varepsilon w^T (B\xi) + \varepsilon^2 w^T B w$$

$$= \lambda + 2\varepsilon w^T B \xi + \varepsilon^2 w^2 B w$$

and

$$\lambda v^T v = \lambda (\xi + \varepsilon w)^T (\xi + \varepsilon w)$$

$$= \lambda(\xi^T \xi + 2\varepsilon w^T \xi + \varepsilon^2 w^T w)$$

$$= \lambda + 2\varepsilon\lambda w^T \xi + \varepsilon^2 \lambda w^T w$$

#7(iii) continues...

It follows that

$$\cancel{\lambda} + 2\varepsilon w^T B\xi + \varepsilon^2 w^T B w \le \cancel{\lambda} + 2\varepsilon\lambda w^T\xi + \varepsilon^2 \lambda w^T w$$

Therefore canceling, and using the fact that $\varepsilon > 0$ implies

$$2w^T B\xi + \varepsilon w^T B w \le 2\lambda w^T \xi + \varepsilon \lambda w^T w$$

Finally rearrange the terms and factor to obtain

$$2w^T(B\xi - \lambda\xi) \le \varepsilon(\lambda w^T w - w^T B w)$$

which was the desired inequality.

#7(iv) Now set $w = B\xi - \lambda\xi$. Is it true or false that ξ must be an eigenvector of B with λ as an eigenvalue?

Setting $w = B\xi - \lambda\xi$ and substituting into the left side we obtain

$$\|B\xi - \lambda\xi\|^2 \leq \varepsilon\left(\lambda w^T w - w^T B w\right)$$

as this must hold for all $\varepsilon > 0$ we obtain that $\|B\xi - \lambda\xi\| = 0$ and consequently that

$$B\xi = \lambda\xi.$$

Noting that $\xi \neq 0$ because ξ is a unit vector immediatly proves that ξ is an eigenvector of B with λ as the corresponding eigenvalue.