Math 466/666: Midterm Version 1

This quiz is open book, open web-browser and open notes. Starting now until after the due date on November 17 do not send email, text or any other type of message to anyone about questions appearing on this quiz. You may send a message to the instructor through WebCampus if you have a question or find an error in the quiz.

Work each problem using pencil and paper or using a computer and Julia as appropriate. Include all work, programs and computer output used to arrive at your final answers. When you are finished upload a high-resolution version of your work as a single PDF file for grading to WebCampus.

**1.** Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this quiz" followed by your signature as the answer to this question.

**2.** Let $x = g(y)$ be the function inverse to $y = f(x)$.

**(i)** Show using induction that one can write the $k$-th derivative as

$$g^{(k)}(y) = \frac{X_k}{(y')^{2k-1}} \qquad \text{for} \qquad k = 1, 2, \ldots$$

where $X_k$ is a polynomial in $y', y'', \ldots, y^{(k)}$ which satisfies the recurrence relation

$$X_{n+1} = \frac{dX_n}{dx} y' - (2n-1) X_n y'' \quad \text{where} \quad X_1 = 1, \quad \text{for} \quad n = 1, 2, \ldots.$$

**(ii)** Use this result to find explicit expressions for $g^{(k)}(y)$ for $k = 1, 2, 3$.

**3.** Given $f\colon \mathbf{R} \to \mathbf{R}$ and the points $x_j = x_1 + h(j-1)$ for $h > 0$, consider the Lagrange interpolating polynomial

$$p(t) = \sum_{j=1}^{n} f(x_j)\ell_j(t) \qquad \text{where} \qquad \ell_j(t) = \prod_{i \neq j} \frac{t - x_i}{x_j - x_i}.$$

**(i)** For $n = 3$ find a bound on $h^3 f'''(t)$ which assures

$$|p(t) - f(t)| \le 10^{-d} \qquad \text{for all} \qquad t \in [x_1, x_3].$$

**(ii)** for $n = 5$ find a similar bound on $h^5 f^{(5)}(t)$ for $t \in [x_1, x_5]$.

**(iii)** Use these results to estimate the maximum value of $h$ in both the three and five point cases to interpolate $\sin t$ on $[-\pi, \pi]$ with an error of less than $10^{-9}$.

**4.** Let $A \in \mathbf{R}^{m \times n}$ with $m > n$ be a matrix with linearly independent columns. This means that the only solution to $Ax = 0$ is $x = 0$. Show that the matrix $G = A^T A$ is square, symmetric and guaranteed to be invertible.

**5.** A matrix $A \in \mathbf{R}^{n\times n}$ with entries $a_{ij}$ is said to strictly diagonally dominant if

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}| \qquad \text{for} \qquad i = 1, \ldots, n.$$

Let $D = \texttt{Diagonal}(A)$ be the diagonal part of $A$. Thus, $D$ has $a_{ii}$ on its diagonal and zeros everywhere else. Define $B = -D^{-1}(A - D)$.

**(i)** Show $\|B\|_\infty < 1$ and conclude the Jacobi iteration for solving $Ax = b$ converges.

**(ii)** Is it true $A$ must be invertible? If so explain why, if not provide a counter example.

**6.** Consider the matrix $A \in \mathbf{R}^{12\times 3}$ with entries $a_{ij}$ chosen randomly (and independently) from the uniform distribution on the interval $[-1, 1]$. In Julia this means

$$A = 2 * \texttt{rand}(12, 3)\ \texttt{.}- 1.$$

Since the columns of $A$ will, in general, be independent then $G = A^T A$ will generally be invertible. The goal for this problem is to experimentally determine the probability that the matrix $G$ is actually diagonally dominant.

**(i)** Write a program that randomly creates a matrix $A$ according to the distribution described above and then checks whether $G = A^T A$ is diagonally dominant.

**(ii)** Add a loop to the above program to create 100 different random matrices $G$ and then compute the proportion of how many were diagonally dominant.

**(iii)** Repeat the above question using samples of 1000 and if possible 10000 matrices. Comment on whether your results are consistent and their accuracy.

**(iv)** [Extra Credit] Repeat the previous steps, except rather than estimating the probably that $G$ is diagonally dominant, instead estimate the probably that the Jacobi iteration for solving $Gx = b$ converges.

#1. I have worked independently on this quiz.

— Test Student.

#2. Let $x = g(y)$ be the inverse to $y = f(x)$.

(i) $$g'(y) = \frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{f'(x)} = \frac{1}{y'} .$$

$$g''(y) = \frac{d}{dy}\frac{1}{y'} = \frac{-1}{(y')^2}\frac{d}{dy}y'$$

$$= \frac{-1}{(y')^2}\frac{d}{dx}y'\frac{dx}{dy} = \frac{-1}{(y')^3}y''$$

In general, if $g^{(n)}(y) = \frac{X_n}{(y')^{2n-1}}$ then

$$g^{(n+1)}(y) = \frac{d}{dy}g^{(n)}(y) = \frac{d}{dy}\frac{X_n}{(y')^{2n-1}}$$

$$= \frac{\left(\frac{d}{dy}X_n\right)(y')^{2n-1} - X_n\frac{d}{dy}(y')^{2n-1}}{(y')^{4n-2}}$$

$$= \frac{\left(\frac{d}{dx}X_n\right)\frac{dx}{dy}(y')^{2n-1} - X_n\frac{d}{dx}(y')^{2n-1}\frac{dx}{dy}}{(y')^{4n-2}}$$

$$= \frac{\frac{dX_n}{dx}\,\frac{1}{y'}(y')^{2n-1} - X_n(2n-1)(y')^{2n-2}\,y''\,\frac{1}{y'}}{(y')^{\cancel{4n-2}\;2n}}$$

$$= \frac{\frac{dX_n}{dx} - (2n-1)X_n\,y''\,\frac{1}{y'}}{(y')^{2n}} \cdot \frac{y'}{y'}$$

$$= \frac{\frac{dX_n}{dx}\,y' - (2n-1)X_n\,y''}{(y')^{2n+1}} = \frac{X_{n+1}}{(y')^{2n+1}}$$

Therefore

$$X_{n+1} = \frac{dX_n}{dx}\,y' - (2n-1)X_n\,y''$$

and from the fact that $g'(y) = \frac{1}{y'}$ we have

$$X_1 = 1.$$

#2 (i) Use the recurrence to find $g^{(k)}(y)$ for $k=1,2,3$.

Since $X_1 = 1$ then $k=1$ obtains

$$g'(y) = \frac{X_1}{(y')^{2\cdot1-1}} = \frac{1}{y'}$$

Now

$$X_2 = \frac{dX_1}{dx} y' - X_1 y'' = -y''$$

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consequently

$$g''(y) = \frac{X_2}{(y')^{2\cdot2-1}} = \frac{-y''}{(y')^3}.$$

Finally

$$X_3 = \frac{dX_2}{dx} y' - (2\cdot2-1) X_2 y''$$

$$= \left[\frac{d}{dx}(-y'')\right] y' - 3(-y'')y''$$

$$= -y'''y' + 3(y'')^2$$

Therefore

$$g'''(x) = \frac{X_3}{(y')^{2\cdot3-1}} = \frac{-y'''y' + 3(y'')^2}{(y')^5}.$$

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#3. Given $f: \mathbb{R} \to \mathbb{R}$ and the points $x_j = x_1 + h(j-1)$ for $h > 0$, consider the Lagrange interpolating polynomial

$$p(t) = \sum_{j=1}^{n} f(x_j)\, l_j(t) \quad \text{where } l_j(t) = \prod_{i \neq j} \frac{t - x_i}{x_j - x_i}.$$

(i) For $n=3$ find a bound on $h^3 f'''(t)$ which assures

$$|p(t) - f(t)| \le 10^{-d} \text{ for all } t \in [x_1, x_3].$$

By the Theorem on Interpolating Polynomials

$$|p(t) - f(t)| = \left| \frac{(t-x_1)(t-x_2)(t-x_3)}{3!} f'''(\xi) \right|$$

for some $\xi$ between $\min(t, x_1)$ and $\max(t, x_3)$. Since $t \in [x_1, x_3]$, then we may assume $\xi \in [x_1, x_3]$.

By definition $x_j = x_1 + h(j-1)$. Therefore

$$|t - x_i| \le |x_3 - x_1| = 2h \text{ for } i = 1, 2, 3.$$

It follows that

$$|p(t) - f(t)| \le \frac{8h^3}{6} |f'''(\xi)| = \frac{4}{3} h^3 |f'''(\xi)|.$$

Consequently, taking

$$h^3 |f'''(\xi)| \le \frac{3}{4} \times 10^{-d}$$

ensures that

$$|p(t) - f(t)| \le 10^{-d} \text{ for all } t \in [x_1, x_3].$$

#3 (ii) For $n=5$ find a similar bound on $h^5 f^{(5)}(t)$ for $t \in [x_1, x_5]$.

Everything is the same, except $n=5$. Thus

$$|p(t)-f(t)| = \left| \frac{\prod_{i=1}^{5}(t-x_i)}{5!} f^{(5)}(\xi) \right|$$

and

$$|t-x_i| \le |x_5 - x_1| = 4h \quad \text{for } i=1,\dots,5.$$

It follows that

$$|p(t)-f(t)| \le \frac{4^5 h^5}{5!} |f^{(5)}(\xi)| = \frac{128}{15} |f^{(5)}(\xi)|.$$

Consequently, taking

$$h^5 |f^{(5)}(\xi)| \le \frac{15}{128} \times 10^{-d}$$

ensures that

$$|p(t)-f(t)| \le 10^{-d} \text{ for all } t \in [x_1, x_5].$$

(iii) Use these results to estimate the maximum value of $h$ in both the three and five point cases to interpolate $\sin t$ on $[-\pi, \pi]$ with an error of less than $10^{-9}$.

Let $f(t) = \sin t$.

Since the bound on $f^{(3)}(t)$ and $f^{(5)}(t)$ is 1 in both cases then

for $n=3$ we need that

$$h^3 \leq \frac{3}{4} 10^{-9}$$

or that

$$h \leq \sqrt[3]{\frac{3}{4} 10^{-9}} = 0.000908560296$$

For $n=5$ we need that

$$h^5 \leq \frac{15}{128} \times 10^{-9}$$

or that

$$h \leq \sqrt[5]{\frac{15}{128} \times 10^{-9}} = 0.0103222 9479$$

#4 Let $A \in \mathbb{R}^{m \times n}$ with $m > n$ be a matrix with linearly independent columns. This means that the only solution to $Ax = 0$ is $x = 0$. Show that the matrix $G = A^T A$ is square, symmetric and guaranteed to be invertible.

<u>Square</u>:

$$G = \underset{n \times m}{A^T} \underset{m \times n}{A} \qquad \text{so } G \in \mathbb{R}^{n \times n}.$$
$$n \times n$$

<u>Symmetric</u>:

$$G^T = (A^T A)^T = A^T A^{TT} = A^T A = G$$

Therefore $G$ is symmetric.

<u>Invertible</u>: Since a square matrix with linearly independent columns must be invertible, then it is sufficient to show that $G$ has linearly independent columns.

Equivalently, we must show that $Gx = 0$ implies that $x = 0$.

Suppose $Gx = 0$. Then $A^T A x = 0$. and so $x^T A^T A x = 0$, which means $\|Ax\|^2 = 0$. Thus $Ax = 0$ and since $A$ has linearly independent columns it follows that $x = 0$ as was to be shown.

#5. A matrix $A \in \mathbb{R}^{n \times n}$ with entries $a_{ij}$ is said to be diagonally dominant if

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad \text{for } i = 1, \dots, n.$$

Let $D$ be the diagonal part of $A$. Thus, $D$ has $a_{ii}$ on its diagonal and zeros everywhere else. Define $B = -D^{-1}(A-D)$.

(i) Show that $\|B\|_\infty < 1$ and conclude the Jacobi iteration for solving $Ax = b$ converges.

$$\|B\|_\infty = \max \left\{ \sum_{j=1}^{n} |B_{kj}| : k = 1, \dots, n \right\}$$

$$B_{kj} = -[D^{-1}(A-D)]_{kj}$$

$$= -\sum_{l=1}^{n} D^{-1}_{kl} (A-D)_{lj}$$

Since $D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & & 0 \\ & \frac{1}{a_{22}} \ddots & \\ 0 & & \frac{1}{a_{nn}} \end{bmatrix}$ is also a diagonal matrix, then the only terms which survive in the sum are where $k = l$. Thus

$$B_{kj} = -D^{-1}_{kk}(A-D)_{kj} = \frac{-1}{a_{kk}}(A-D)_{kj}$$

Now

$$A-D=\begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n} \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{bmatrix}$$

is the same a $A$ except with zeros on the diagonal, then

$$\sum_{j=1}^{n}|B_{kj}| = \sum_{j\neq k} \frac{1}{|a_{kk}|}|a_{kj}| = \frac{1}{|a_{kk}|}\sum_{j\neq k}|a_{kj}|$$

Since the definition of diagonally dominant implies

$$|a_{kk}| > \sum_{j\neq k}|a_{kj}| \quad \text{for } k=1,\dots,n$$

then $\sum_{j=1}^{n}|B_{kj}| < 1$ for every $k=1,\dots,n$.

Taking the maximum over $n$ such terms yields

$$\|B\|_\infty = \max\left\{\sum_{j=1}^{n}|B_{kj}| : k=1,\dots,n\right\} < 1$$

and therefore the Jacobi method converges.

#5(ii) Is it true $A$ must be invertible? If so explain why, if not provide a counter example.

To see that $A$ is invertible we need to show that $Ax = b$ has a solution for every $b \in \mathbb{R}^n$ and that this solution is unique.

<u>Existence of solution</u>: The existence of $x \in \mathbb{R}^n$ follows immediately from the fact that the Jacobi iterations converge, because the limit is a solution such that $Ax = b$.

<u>Uniqueness of solution</u>: Suppose there were two solutions $x$ and $y$ such that

$$Ax = b \quad \text{and} \quad Ay = b$$

Then $A(x-y) = 0$. Define $z = x - y$. We claim that $Az = 0$ implies that $z = 0$. Suppose $z \neq 0$ and let $k$ be choosen so that

$$|z_i| \leq |z_k| \quad \text{for every } i = 1, \ldots, n.$$

Thus

$$|z_k| = \|z\|_\infty = \max\{|z_i| : i = 1, \ldots, n\}.$$

Then

$$(Az)_k = \sum_{j=1}^{n} a_{kj} z_j$$

$$= a_{kk} z_k + \sum_{j \neq k} a_{kj} z_j$$

implies

$$|(Az)_k| \geq |a_{kk}||z_k| - \sum_{j \neq k} |a_{kj}||z_j|$$

$$\geq |a_{kk}||z_k| - |z_k| \sum_{j \neq k} |a_{kj}|$$

$$= |z_k| \left( |a_{kk}| - \sum_{j \neq k} |a_{kj}| \right) > 0.$$

Therefore $Az \neq 0$. This contradicts the assumption that $Az = 0$. Therefore it must be that $z = 0$. This means $x = y$ and thus the solutions are unique.

# Question6

December 1, 2020

### Midterm Problem 6

Consider the matrix $A \in \mathbf{R}^{12\times 3}$ with entries $a_{ij}$ chosen randomly and independently from the uniform distribution on the interval $[-1, 1]$. Since the columns of $A$ will, in general, be independent then $G = A^T A$ will generally be invertible. The goal for this problem is to experimentally determine the probability that the matrix $G$ is actually diagonally dominant.

(i) Write a program that randomly creates a matrix $A$ according to the distribution described above and then checks whether $G = A^T A$ is diagonally dominant.

[1]:
```
A=2*rand(12,3).-1
```

[1]:
```
12×3 Array{Float64,2}:
 -0.193198   -0.945849    -0.76916
  0.529891   -0.272145    -0.97467
  0.434714    0.00952433   0.993245
 -0.349979    0.0611721    0.297463
  0.889205    0.359605     0.323422
 -0.870304    0.562668     0.0713288
 -0.261916    0.741196     0.84753
 -0.07264    -0.461116    -0.00724868
 -0.270022    0.290023    -0.478484
 -0.541424    0.446682     0.516281
 -0.300788   -0.0632127   -0.277333
  0.0507238   0.528206     0.732719
```

[2]:
```
G=A'*A
```

[2]:
```
3×3 Array{Float64,2}:
  2.71066    -0.583655  -0.0658825
 -0.583655    2.74707    2.30478
 -0.0658825   2.30478    4.55394
```

[3]:
```
function isdd(a)
    m,n=size(a)
    r=0
    for i=1:m
        r=abs(a[i,i])
        for j=1:n
```

```
            if j!=i
                r=r-abs(a[i,j])
            end
        end
    end
    return r>0
end
```

[3]: isdd (generic function with 1 method)

[4]:
```
isdd(G)
```

[4]: true

(ii) Add a loop to the above program to create 100 different random matrices $g$ and then compute the proportion of how many were diagonally dominant.

[9]:
```
count=0
for k=1:100
    A=2*rand(12,3).-1
    G=A'*A
    if isdd(G)
        count=count+1
    end
end
println("A total of $count out of 100 were diagonally dominant")
```

```
A total of 97 out of 100 were diagonally dominant
```

(iii) Repeat the above question using samples of 1000 and if possible 10000 matrices. Comment on whether your results and consistent and their accuracy.

[10]:
```
function trials(total)
    count=0
    for k=1:total
        A=2*rand(12,3).-1
        G=A'*A
        if isdd(G)
            count=count+1
        end
    end
    return count
end
```

[10]: trials (generic function with 1 method)

[12]:
```
trials(1000)
```

[12]: 964

[13]:
```
trials(10000)
```

[13]: 9613

Since the proportion of 964 to 1000 is about the same as 9613 to 10000, the results appear consistent. To get an idea of the accuracy, we repeat the 10000 trials result a number of times to see what different answers are obtained.

[16]:
```
for j=1:20
    println("A total of ",trials(10000),
        " out of 10000 were diagonally dominant")
end
```

```
A total of 9612 out of 10000 were diagonally dominant
A total of 9608 out of 10000 were diagonally dominant
A total of 9592 out of 10000 were diagonally dominant
A total of 9623 out of 10000 were diagonally dominant
A total of 9641 out of 10000 were diagonally dominant
A total of 9620 out of 10000 were diagonally dominant
A total of 9578 out of 10000 were diagonally dominant
A total of 9582 out of 10000 were diagonally dominant
A total of 9633 out of 10000 were diagonally dominant
A total of 9607 out of 10000 were diagonally dominant
A total of 9620 out of 10000 were diagonally dominant
A total of 9567 out of 10000 were diagonally dominant
A total of 9609 out of 10000 were diagonally dominant
A total of 9587 out of 10000 were diagonally dominant
A total of 9586 out of 10000 were diagonally dominant
A total of 9563 out of 10000 were diagonally dominant
A total of 9611 out of 10000 were diagonally dominant
A total of 9569 out of 10000 were diagonally dominant
A total of 9585 out of 10000 were diagonally dominant
A total of 9593 out of 10000 were diagonally dominant
```

As the result of twenty tests were all within a percent of each other, we comment that the accuracy looks like we know the answer to within about one percent. Note that a more detailed statistical analysis is possible, but outside the scope of this course.

(iv) Repeat the previous steps, except rather than estimating the probability that $G$ is diagonally dominant, instead estimate the probability that the Jacobi iteration for solving $Gx = b$ converges.

To check that the Jacobi iteration converges we actually compute the iteration and look to see if successive iterates get close together.

[44]:
```
function jacobi(a,b,x)
    n=length(x)
```

```
    xt=copy(b)
    for i=1:n
        for j=1:n
            if i!=j
                xt[i]=xt[i]-a[i,j]*x[j]
            end
        end
        xt[i]=xt[i]/a[i,i]
    end
    return xt
end
```

[44]: jacobi (generic function with 3 methods)

[49]:
```
using LinearAlgebra
```

If after 10000 iterations the norm of successive iterates is less than $10^{-8}$ we conclude they converged; otherwise, not.

[68]:
```
function isconv(a)
    m,n=size(a)
    x=rand(n)
    b=rand(m)
    y=x
    for k=1:10000
        y=x
        x=jacobi(a,b,y)
    end
    return norm(x-y)<1e-8
end
```

[68]: isconv (generic function with 1 method)

The function `etrials` is the same as `trials` except that it also called the `isconv` function to check whether the Jacobi iterations converged or not. The return result is the ordered pair of number of diagonally dominant matrices followed by how many converged.

[69]:
```
function etrials(total)
    count=0
    conv=0
    for k=1:total
        A=2*rand(12,3).-1
        G=A'*A
        if isdd(G)
            count=count+1
        end
        if isconv(G)
            conv=conv+1
```

```
        end
    end
    return count,conv
end
```

[69]: etrials (generic function with 1 method)

[70]:
```
etrials(10000)
```

[70]: (9580, 9869)

The Jacobi iteration did not appear to converge in all cases, but there were many examples that were not diagonally domininant and for which the iteration converged.

[ ]:
```

```