Math 466/666: Midterm Version A

This is a closed-book closed-notes no-calculator-allowed in-class exam. Efforts have been made to keep the arithmetic simple. If it turns out to be complicated, that's either because I made a mistake or you did. In either case, do the best you can and check your work where possible. While getting the right answer is nice, this is not an arithmetic test. It's more important to clearly explain what you did and what you know.

1. Indicate in writing that you have understood the requirement to work independently by writing "I have worked independently on this exam" followed by your signature as the answer to this question.

I have worked independently on this exam

— Your Name

2. Consider the matrix A with inverse A^{-1} given by

$$A = \begin{bmatrix} -1 & -2 & 0 & 0 \\ 1 & -1 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad A^{-1} = \begin{bmatrix} -\frac{1}{3} & 0 & -\frac{4}{3} & -\frac{4}{3} \\ -\frac{1}{3} & 0 & \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{6} & -2 & -\frac{11}{3} & -\frac{14}{3} \\ -\frac{1}{6} & 0 & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}.$$

$2\frac{1}{2} \quad 3\frac{1}{2} \quad \frac{1}{2} \quad 2\frac{1}{2}$

$1 \quad 2 \quad \frac{18}{3} \quad \frac{22}{3}$

Recall that the matrix 1-norm of A may be computed as

$$\|A\|_1 = \max\left\{ \sum_{i=1}^{n} |A_{ik}| : k = 1, \ldots, n \right\} \qquad \text{where} \qquad n = 4.$$

^sum of abs values along columns

Compute $\|A\|_1$ and $\|A^{-1}\|_1$ and then use the 1-norm to find $\operatorname{cond}(A)$.

$\|A\|_1 = \max\{2\frac{1}{2}, 3\frac{1}{2}, \frac{1}{2}, 2\frac{1}{2}\} = 3\frac{1}{2} = \frac{7}{2}$

$\|A^{-1}\|_1 = \max\{1, 2, \frac{18}{3}, \frac{22}{3}\} = \frac{22}{3}$

$\operatorname{Cond}(A) = \|A\|_1 \|A^{-1}\|_1 = \frac{7}{2} \cdot \frac{22}{3} = \frac{77}{3}$

3. Let $f\colon \mathbf{R}\to\mathbf{R}$ be a twice continuously differentiable function such that $f(\alpha)=0$ and $f'(\alpha)\neq 0$. Explain why Newton's method, given by

$$x_{n+1}=x_n-\frac{f(x)}{f'(x)}\qquad\text{where}\qquad x_0 \text{ is an initial approximation}$$

is quadratically convergent. That is, provided x_0 is sufficiently close to α prove there exists a constant M such that

$$|e_{n+1}|\le M|e_n|^2\qquad\text{where}\qquad e_n=x_n-\alpha\qquad\text{for}\qquad n=0,1,2,\ldots.$$

By Taylor's theorem

$$0=f(\alpha)=f(x_n-e_n)=f(x_n)-e_nf'(x_n)+\frac{e_n^2}{2}f''(\xi_n)$$

where ξ_n is between x_n and α.

Thus, dividing by $f'(x_n)$ yields

$$0=\frac{f(x_n)}{f'(x_n)}-e_n+\frac{e_n^2}{2}\frac{f''(\xi_n)}{f'}$$

def of e_{n+1} Newton's method

$$e_{n+1}=x_{n+1}-\alpha=x_n-\frac{f(x_n)}{f'(x_n)}-\alpha=e_n-\frac{f(x_n)}{f'(x_n)}$$

by Taylor

$$=\cancel{e_n}-\left(\cancel{e_n}-\frac{e_n^2}{2}\frac{f''(\xi_n)}{f'(x_n)}\right)=\frac{f''(\xi_n)}{2f'(x_n)}e_n^2$$

The only thing left is to bound by a constant... → continues on the back page →

4. Explain why it is sometimes said the number of correct significant digits approximately doubles with each iteration when using Newton's method.

We have $|e_{n+1}|\le M|e_n|^2$ Suppose $|e_n|\approx 10^{-k}$ then $|e_{n+1}|\le M(10^{-k})^2=M\cdot 10^{-2k}$ Suppose $M=10^p$ then $|e_{n+1}|\le 10^{p-2k}$ if k is very big compared to p then $p-2k\approx -2k$ and k will be very big after the sequence has iterated for some time.

Proof of quadratic convergence of Newton's method continues...

Since $f'(\alpha) \neq 0$ there is a δ-neighborhood of α such that $f'(x) \neq 0$ for all $|x-\alpha| \leq \delta$.

define $B = \min\{ |f'(x)| : |x-\alpha| \leq \delta \} > 0$

For the same δ define

$$A = \max\{ |f''(x)| : |x-\alpha| \leq \delta \} < \infty$$

Therefore if $|x_n - \alpha| \leq \delta$ and $|\xi_n - \alpha| \leq \delta$ then

$$|e_{n+1}| \leq \frac{A}{2B} |e_n|^2$$

Need to make sure that $|e_{n+1}| < |e_n|$ so that if $|x_0 - \alpha| \leq \delta$ then also $|x_1 - \alpha| \leq \delta$ and so forth...

To do this note that

$$|e_{n+1}| \leq \boxed{\frac{A}{2B} |e_n|} \, |e_n|$$

need this less than 1 so the estimate can propagate forward...

Need

$$\frac{A}{2B} |e_0| < 1$$

so if x_0 is close enough to α such that

$$|x_0 - \alpha| < \min\left(\delta, \frac{2B}{A}\right)$$

then the Newton's method converges quadratically...

5. Given a unit vector $v \in \mathbf{R}^n$ the corresponding Hausholder reflector is $H = I - 2vv^T$. Show that H is an orthogonal matrix.

$$H^T = (I - 2vv^T)^T = I^T - (2vv^T)^T$$

$$= I^T - 2\, v^{T^T} v^T = I - 2vv^T$$

Thus $H^T = H$ and H is symmetric...

$$H^T H = (I - 2vv^T)(I - 2vv^T) = I - 2vv^T - 2vv^T + 4v(v^T v)v^T$$

$$= I - 4vv^T + 4vv^T = I$$

6. Factor the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad r_2 \leftarrow r_2 - 3r_1 \quad \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

as $A = LU$ where L is lower triangular and U is upper triangular.

Thus $U = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

7. Consider a hypothetical computer which performs all arithmetic in decimal using round-to-the-nearest with 7 significant digits. Let $x = 1.437592$ and $y = 1.431256$. Find the value of $z = x - y$ as calculated by the computer and explain how many digits of precision were lost when performing the calculation.

$$\begin{array}{r} 1.437592 \\ 1.431256 \\ \hline \end{array}$$

$$z = \underbrace{0.00}6336$$

3 digits cancelled so 3 digits of precision were lost.

8. If you are enrolled in Math 466 prove one of the following theorems; if you are enrolled in Math 666 prove both of them:

Proposition 6.3 *All eigenvalues of Hermitian matrices are real.*

Proposition 6.4 *Eigenvectors corresponding to distinct eigenvalues of Hermitian matrices must be orthogonal.*

Proof of Proposition 6.3

Let λ be an eigenvalue with eigenvector x

Thus $\boxed{Ax = \lambda x.}$ λ real means $\lambda = \bar{\lambda}$.

$$\bar{x} \cdot Ax = Ax \cdot \bar{x} = (Ax)^T \bar{x}$$
$$= x^T A^T \bar{x} = x^T \overline{A^H} \bar{x}$$
$$= x^T \overline{Ax} = x \cdot \overline{Ax}$$

Thus

$$\bar{x} \cdot Ax = x \cdot \overline{Ax}$$
$$\bar{x} \cdot \lambda x = x \cdot \overline{\lambda x}$$
$$\lambda \cancel{\bar{x} \cdot x} = \bar{\lambda}\, \cancel{x \cdot \bar{x}}$$
$$\lambda = \bar{\lambda}$$

So λ is real...

since $x \in \mathbb{C}^n$ is not zero, then $x \cdot \bar{x} > 0$

$\|x\|^2$

Proof of Prop 6.4 is on the back

Proof of Prop 6.4

Let $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$

and $A^H = A$. Note from Prop. 6.3 $\lambda_1, \lambda_2 \in \mathbb{R}$

Need to show: $x_1 \cdot \bar{x}_2 = 0$ since $x_1, x_2 \in \mathbb{C}^n$

$$\bar{x}_1 \cdot Ax_2 = Ax_2 \cdot \bar{x}_1 = (Ax_2)^T \bar{x}_1$$

$$= x_2^T A^T \bar{x}_1 = x_2^T \overline{A^H} \bar{x}_1$$

$$= x_2^T \overline{Ax_1} = x_2 \cdot \overline{Ax_1}$$

Therefore

$$\bar{x}_1 \cdot Ax_2 = x_2 \cdot \overline{Ax_1}$$

$$\bar{x}_1 \cdot \lambda_2 x_2 = x_2 \cdot \overline{\lambda_1 x_1}$$

$$\lambda_2 \bar{x}_1 \cdot x_2 = \bar{\lambda}_1 x_2 \cdot \bar{x}_1$$

since $\lambda_1 \in \mathbb{R}$ then $\bar{\lambda}_1 = \lambda_1$

$$\lambda_2 \bar{x}_1 \cdot x_2 = \lambda_1 x_2 \cdot \bar{x}_1$$

Thus,

$$(\lambda_2 - \lambda_1) \bar{x}_1 \cdot x_2 = 0$$

Since $\lambda_1 \neq \lambda_2$ then $\bar{x}_1 \cdot x_2 = 0$ ... ✓