# Polynomial Interpolation

Two points determine a line uniquely.

Three points determine a parabola uniquely.

⋮ ⋮

$n$ points determine a $(n-1)$-degree polynomial

$$p_{n-1}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}$$

$n$ parameters determined by $n$ points.

In particular given $n$ points

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

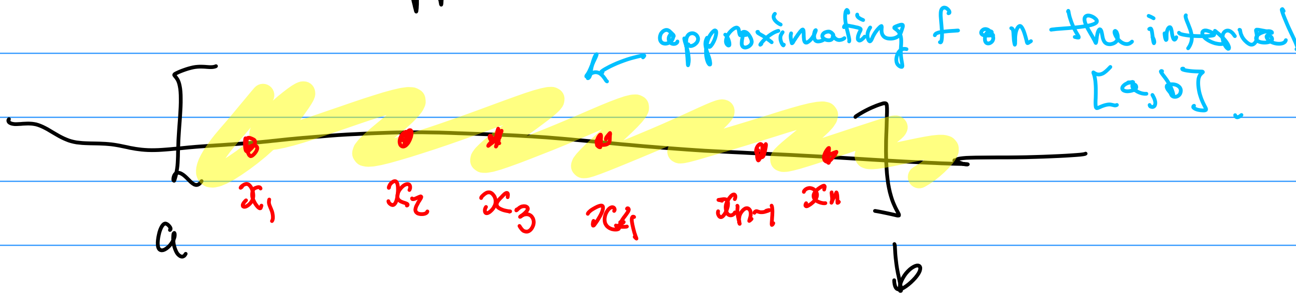
there is a unique polynomial $p$ of degree $n-1$ such that

$$y_1 = p(x_1), \quad y_2 = p(x_2), \quad \ldots, \quad y_n = p(x_n)$$

Polynomials a building blocks that can be used to approximate other functions...

Let's suppose $f$ is a function we want to approximate...

Sample the function at some points near where we want an appoximation




Define

$$y_1 = f(x_1), \quad y_2 = f(x_2) \quad \cdots \quad y_n = f(x_n)$$

to obtain $n$ points

$$(x_1, y_1), \quad (x_2, y_2), \quad \ldots \quad (x_n, y_n)$$

Now let $p$ be the polynomial of degree $n-1$ that passes through these points...

Question how close are $p$ and $f$ ?

Let $\underbrace{E(x) = f(x) - p(x)}$. How big is $E$ ?

Idea look for an expression that characterizes $E$ in terms of $f$ and the points $x_1, x_2, \ldots, x_n$ that can then be bounded in estimates...

Note $E$ has $n$ zeros. That is,

$$E(x_1) = 0, \quad E(x_2) = 0, \quad \ldots, \quad E(x_n) = 0$$

Theorem: Let $p(t)$ be the interpolating polynomial of degree $n-1$ such that $p(x_i) = f(x_i)$ for $i=1,\dots,n$. Assume the $x_i$'s are different, also that $f$ has $n$ continuous derivatives. Then

$$f(t) = \underbrace{p(t)}_{\text{interpolating polynomial}} + \frac{q(t)}{n!} f^{(n)}(\xi)$$

another polynomial

where

$$q(t) = (t-x_1)(t-x_2)\cdots(t-x_n)$$

and $\xi$ is between $\min(t, x_1, x_2, \dots, x_n)$ and $\max(t, x_1, x_2, \dots, x_n)$.

Compare with Taylor theorem...

Suppose $f$ is a function with $n$ (continuous) derivatives

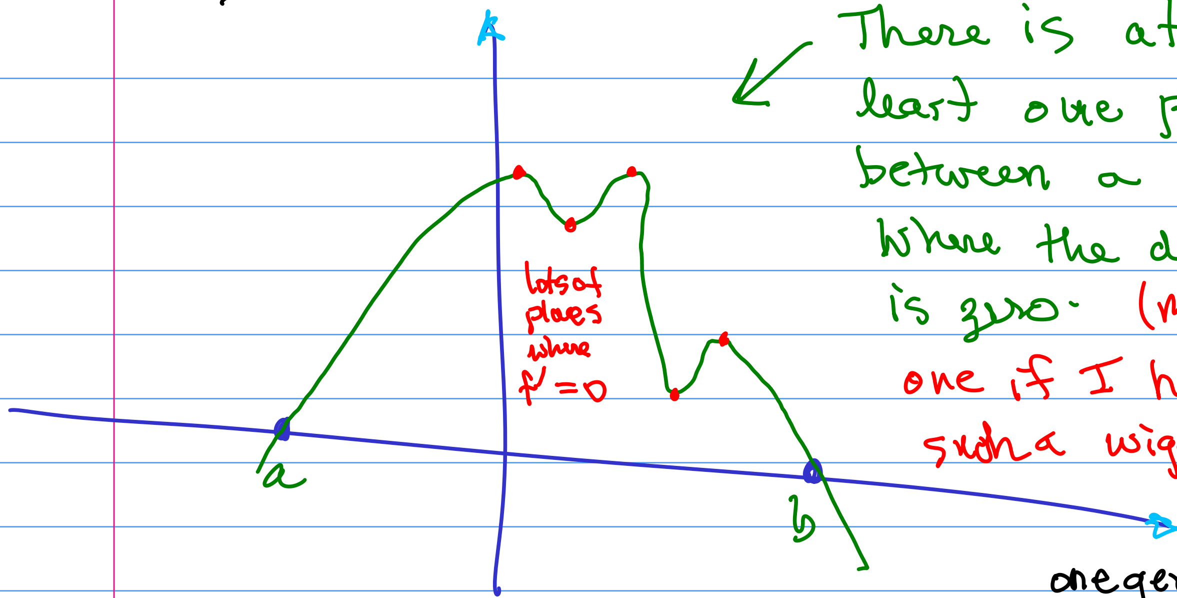
$$f(t) = f(a) + (t-a)f'(a) + \frac{(t-a)^2}{2!} f''(a) + \cdots + \frac{(t-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

where

$$R_n = \frac{(t-a)^n}{n!} f^{(n)}(\xi) \quad \text{for some } \xi \text{ between } t \text{ and } a.$$

There is a proof of Taylor's theorem (and also the interpolating polynomial theorem) based on Rolle's Theorem. What's that?

## Rolle's Theorem:

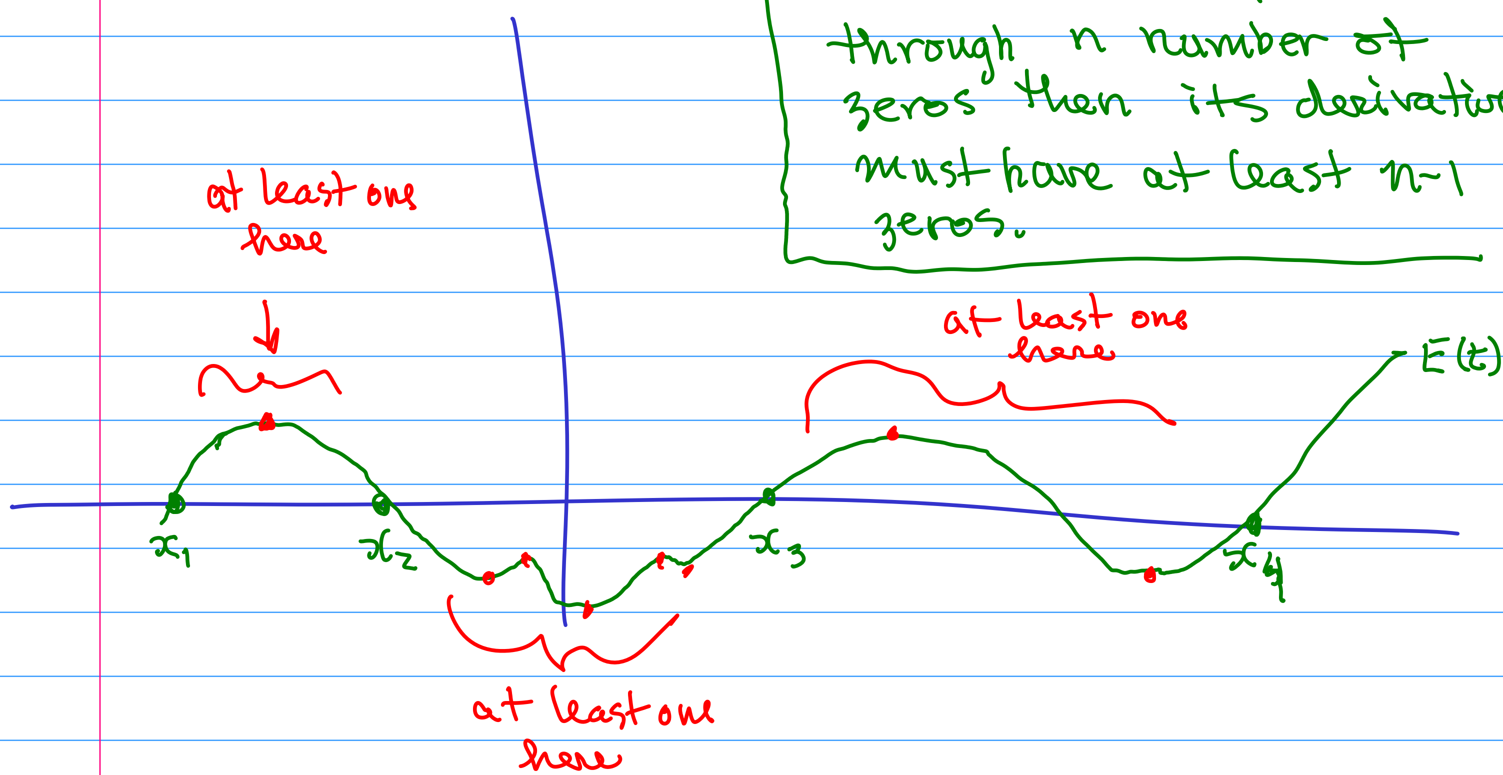



There is at least one point between a and b where the derivative is zero. (maybe only one if I hadn't drawn such a wiggly line)

one generalization is the mean value theorem...

## Generalize Rolle's theorem

If the function passes through n number of zeros then its derivative must have at least n-1 zeros.




Note that $g(x_1)=0,\ g(x_2)=0,\ \ldots,\ f(x_n)=0$

$E(x_1)=0,\ E(x_2)=0,\ \ldots,\ E(x_n)=0$

Define, $F(t) = E(t) - \alpha q(t)$

No matter what $\alpha$ is we also have

$$F(x_1)=0, \quad F(x_2)=0, \quad \dots, \quad F(x_n)=0.$$

$\boxed{\checkmark}$ $\boxed{\checkmark}$ ..... $\boxed{\checkmark}$

Let pick a value $t_*$ such that $t_* \neq x_i$ for any of the $x$'s.

$$F(t_*) = E(t_*) - \alpha q(t_*)$$

Setting $\alpha = \dfrac{E(t_*)}{q(t_*)}$ note since $t_* \neq x_i$ for all $i$, then $q(t_*) \neq 0$.

Yield that $F(t_*) = 0$. $\boxed{\checkmark}$

With this choice of $\alpha$ then $F(t)$ has $n+1$ roots

By Rolle's theorem

$F'(t) = E'(t) - \alpha q'(t)$ has $n$ roots

$F''(t) = E''(t) - \alpha q''(t)$ has $n-1$ roots

$\vdots$

$F^{(n)}(t) = E^{(n)}(t) - \alpha q^{(n)}(t)$ has 1 root

Recall,

$q(t) = (t-x_1)(t-x_2)\cdots(t-x_n) = t^n +$ lower order terms

$q^{(n)}(t) = n!$

$F^{(n)}(t) = E^{(n)}(t) - \alpha n!$ has 1 root

Let that root be $\xi$. Then

$$F^{(n)}(\xi) = E^{n}(\xi) - \alpha n! = 0$$

or $E^{n}(\xi) = \alpha n! = \boxed{\dfrac{E(t_*)}{q(t_*)}} n!$

Since

$E(x) = f(x) - p(x)$

$$\alpha = \frac{E(t_*)}{q(t_*)}$$

solve for $E(t_*)$

Then... we'll back up and finish this next time...