Analysis of inverse interpolation method for finding solutions to $f(x)=0$

Theorem: Let $p(t)$ be the interpolating polynomial of degree $n-1$ such that $p(x_i) = f(x_i)$ for $i=1,\dots,n$. Assume the x_i's are different, also that f has n continuous derivatives. Then

$$f(t) = \underbrace{p(t)}_{\text{interpolating polynomial}} + \frac{q(t)}{n!} f^{(n)}(\xi) \quad \text{another polynomial}$$

where

$$q(t) = (t-x_1)(t-x_2)\cdots(t-x_n)$$

and ξ is between $\min(t, x_1, x_2, \dots, x_n)$ and $\max(t, x_1, x_2, \dots, x_n)$.

Rewrite theorem for interpolating the inverse function...

Let $p(t)$ be the interpolating polynomial of degree $n-1$ such that $p(f(x_i)) = x_i$ for $i=1,\dots,n$. Assume the $f(x_i)$'s are different, also that f^{-1} has n continuous derivatives. Then

$$f^{-1}(t) = p(t) + \frac{q(t)}{n!} (f^{-1})^{(n)}(\xi)$$

where

$$q(t) = (t-f(x_1))(t-f(x_2))\cdots(t-f(x_n))$$

and ξ is between $\min(t, f(x_1), \dots, f(x_n))$ and $\max(t, f(x_1), \dots, f(x_n))$.

Let $p_k(t)$ be the interpolating polynomial through the points

$$\underbrace{(f(x_k), x_k), (f(x_{k+1}), x_{k+1}), \dots (f(x_{k+n-1}), x_{k+n-1})}_{n \text{ points...}}$$

Define

$$\boxed{x_{k+n} = p_k(0)} \qquad \text{inverse interpolation step...}$$

note $\alpha = f^{-1}(0)$

Suppose the exact root is α so $f(\alpha) = 0$. Then the error is $e_j = x_j - \alpha$. How does the error in the next approximation depend on the previous errors?

$$e_{k+n} = x_{k+n} - \alpha = \boxed{p_k(0) - f^{-1}(0)} = -\frac{q_k(0)}{n!}(f^{-1})^{(n)}(\xi)$$

where

$$q_k(t) = (t - f(x_k))(t - f(x_{k+1}))\cdots(t - f(x_{k+n-1}))$$

Therefore

since $f(\alpha) = 0$

$$|q_k(0)| = |f(x_k)||f(x_{k+1})|\cdots|f(x_{k+n-1})|$$
$$= |f(x_k) - f(\alpha)||f(x_{k+1}) - f(\alpha)|\cdots|f(x_{k+n-1}) - f(\alpha)|$$
$$= |f'(c_k)\overbrace{(x_k - \alpha)}^{e_k}||f'(c_{k+1})\overbrace{(x_{k+1} - \alpha)}^{e_{k+1}}|\cdots|f'(c_{k+n-1})\overbrace{(x_{k+n-1} - \alpha)}^{e_{k+n-1}}|$$

Suppose $|f'(c)| \leq A$ for all c under consideration...

and also $|(f^{-1})^{(n)}(\xi)| \leq B$ for all ξ under consideration...

$$|e_{k+n}| = \left|\frac{q_k(0)}{n!}(f^{-1})^{(n)}(\xi)\right| \leq \frac{A^n |e_k||e_{k+1}|\cdots|e_{k+n-1}|}{n!} B$$

Example from last time.

$$\boxed{|e_{k+3}| \leq \frac{A^3 B}{3!}|e_k||e_{k+1}||e_{k+2}|}$$

Interpret this in terms of the order γ of the method where

$$|e_{j+1}| \approx M|e_j|^\gamma \quad \text{substitute this in}$$

just constants

Therefore

$$|e_{k+1}| \approx M|e_k|^\gamma$$

$$|e_{k+2}| \approx M|e_{k+1}|^\gamma = M(M|e_k|^\gamma)^\gamma = M^{1+\gamma}|e_k|^{\gamma^2}$$

$$|e_{k+3}| \approx M|e_{k+2}|^\gamma = M(M^{1+\gamma}|e_k|^{\gamma^2})^\gamma = M^{1+\gamma+\gamma^2}|e_k|^{\gamma^3}$$

Thus

$$(\text{Const})|e_k|^{\gamma^3} \leq (\text{Const})|e_k|^1|e_k|^\gamma|e_k|^{\gamma^2}$$

For consistency I need

$$\gamma^3 = 1 + \gamma + \gamma^2$$

Solve this for γ.

To analyse the method for finding roots, I need to find the root of

$$h(\gamma) = \gamma^3 - \gamma^2 - \gamma - 1$$

use Newton's method

$$\gamma_{j+1} = \gamma_j - \frac{\gamma_j^3 - \gamma_j^2 - \gamma_j - 1}{3\gamma_j^2 - 2\gamma_j - 1}$$

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julia> g(x)=x-(x^3-x^2-x-1)/(3*x^2-2*x-1)
g (generic function with 1 method)

julia> x=2
       for j=1:100
           x=g(x)
       end

julia> x
1.8392867552141612
```

order of the method we used last time.

Since $\frac{46}{25} = 1.84$ this is consistent with the numerics...