Proposition 6.2 ([4], Proposition 2.2). Eigenvectors corresponding to different eigenvalues must be linearly independent.

λ_i's with $\lambda_i \neq \lambda_j$ when $i \neq j$ $\quad i = 1, \dots, n$

x_i's such that $A x_i = \lambda_i x_i$

Suppose the x_i were dependent. Then

$$\lambda_2 \left(x_1 c_1 + \cancel{x_2 c_2} + \dots + x_n c_n \right) = 0 \cdot \lambda_2$$

and c_i's are not all zero...

Mult by A

$$A x_1 c_1 + A x_2 c_2 + \dots + A x_n c_n = A 0$$

$$\boxed{x_1 \lambda_1 c_1 + \cancel{x_2 \lambda_2 c_2} + \dots + x_n \lambda_n c_n = 0}$$

$$\lambda_3 \left(x_1 (\lambda_2 - \lambda_1) c_1 + x_3 (\cancel{\lambda_2 - \lambda_3}) c_3 + \dots + x_n (\lambda_2 - \lambda_n) c_n \right) = 0$$

mult by A

$$x_1 (\lambda_2 - \lambda_1) \lambda_1 c_1 + x_3 (\cancel{\lambda_2} - \lambda_3) \lambda_3 c_3 + \dots + x_n (\lambda_2 - \lambda_n) \lambda_n c_n = 0$$

$$x_1 (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) c_1 + x_4 (\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4) c_4 + \dots + x_n (\lambda_2 - \lambda_n)\cancel{(\lambda_3 - \lambda_n)} c_n = 0$$

repeat untill you get

$$x_1 (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \cdots (\lambda_n - \lambda_1) c_1 = 0$$

which means $c_1 = 0$

This same argument shows $c_i = 0$ for every i...

and c_i's are not all zero... $\leftarrow$ contradiction, so the eigenvectors are independent...

Proposition 6.3. All eigenvalues of Hermitian matrices are real.

$A \in \mathbb{C}^{n\times n}$ so that $A^H = A$

$\overline{1+i} = 1-i$,
$$\begin{bmatrix} 1+i & 2+3i \\ 4-2i & 5+i \end{bmatrix}^H = \begin{bmatrix} 1-i & 4+2i \\ 2-3i & 5-i \end{bmatrix}$$

Let λ be an eigenvalue with eigenvector x

Thus $\boxed{Ax = \lambda x.}$ λ real means $\lambda = \bar{\lambda}$.

$$\bar{x}\cdot Ax = Ax\cdot\bar{x} = (Ax)^T\bar{x}$$
$$= x^T A^T \bar{x} = x^T \overline{A^H}\,\bar{x}$$
$$= x^T\,\overline{Ax} = x\cdot\overline{Ax}$$

Thus

$$\bar{x}\cdot Ax = x\cdot\overline{Ax}$$
$$\bar{x}\cdot\lambda x = x\cdot\overline{\lambda x}$$
$$\lambda\,\cancel{\bar{x}\cdot x} = \bar{\lambda}\,\cancel{x\cdot\bar{x}}$$
$$\lambda = \bar{\lambda}$$

since $x \in \mathbb{C}^n$ is not zero, then $x\cdot\bar{x} > 0$

$\|x\|^2$

So λ is real...

Proposition 6.4. Eigenvectors corresponding to distinct eigenvalues of Hermitian matrices must be orthogonal.

Let $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$

and $A^H = A$. Note from prop. 6.3 $\lambda_1, \lambda_2 \in \mathbb{R}$

Need to show: $x_1 \cdot \bar{x}_2 = 0$ since $x_1, x_2 \in \mathbb{C}^n$

$$\bar{x}_1 \cdot Ax_2 = Ax_2 \cdot \bar{x}_1 = (Ax_2)^T \bar{x}_1$$

$$= x_2^T A^T \bar{x}_1 = x_2^T \overline{A^H} \bar{x}_1$$

$$= x_2^T \overline{Ax_1} = x_2 \cdot \overline{Ax_1}$$

Therefore

$$\bar{x}_1 \cdot Ax_2 = x_2 \cdot \overline{Ax_1}$$

$$\bar{x}_1 \cdot \lambda_2 x_2 = x_2 \cdot \overline{\lambda_1 x_1}$$

$$\lambda_2 \bar{x}_1 \cdot x_2 = \bar{\lambda}_1 x_2 \cdot \bar{x}_1$$

since $\lambda_1 \in \mathbb{R}$ then $\bar{\lambda}_1 = \lambda_1$

$$\lambda_2 \bar{x}_1 \cdot x_2 = \lambda_1 x_2 \cdot \bar{x}_1$$

Thus,

$$(\lambda_2 - \lambda_1) \bar{x}_1 \cdot x_2 = 0$$

Since $\lambda_1 \neq \lambda_2$ then $\bar{x}_1 \cdot x_2 = 0$ ... ✓

Theorem 6.1 (Spectral Theorem). Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian (if $A \in \mathbb{R}^{n\times n}$, suppose it is symmetric). Then, A has exactly n orthonormal eigenvectors $\vec{x}_1, \cdots, \vec{x}_n$ with (possibly repeated) eigenvalues $\lambda_1, \ldots, \lambda_n$. In other words, there exists an orthogonal matrix X of eigenvectors and diagonal matrix D of eigenvalues such that $D = X^\top A X$.

By Prop 6.1 from Tuesday every matrix has at least one eigenvector and one eigenvalue.

Since A is Hermitian (symmetric in the real case) it is possible to use the eigenvector and eigenvalue to reduce A to another matrix with lower dimension.

Now apply Prop 6.1 to the smaller matrix and procede by induction.

Note that matrices which are not Hermitian (or symmetric) may have only one eigenvector.

Proposition 6.5. All eigenvalues of positive definite matrices are positive.

Proof. Take $A \in \mathbb{R}^{n\times n}$ positive definite, and suppose $A\vec{x} = \lambda\vec{x}$ with $\|\vec{x}\|_2 = 1$. By positive definiteness, we know $\vec{x}^\top A\vec{x} > 0$. But, $\vec{x}^\top A\vec{x} = \vec{x}^\top(\lambda\vec{x}) = \lambda\|\vec{x}\|_2^2 = \lambda$, as needed. □

$\lambda > 0$

This property is not nearly as remarkable as those associated with symmetric or Her-

by definition positive definite means $x^T A x > 0$

by definition eigenvalue/eigenvector means $Ax = \lambda x$ substitute

result

$x^T \lambda x > 0$

or $\lambda(\cancel{x^T x}) > 0$

thus $\lambda > 0$