[26-Oct-2021] In Class Written Exam

There will be a written exam in class on Tuesday October 26 covering the following topics from the lecture and homework:

- Statement of Newton's method (Aug 26)
- Proof and interpretation of quadraditic convergence (Aug 26)
- Example of loss of precision (Sep 7)
- Backwards error analysis and condition number (Sep 7 and Sec 23)
- Compute matrix 1 and ∞ norms (Sep 21)
- Prove Hausholder reflector is an othogonal matrix (Sep 30)
- Eigenvectors of distinct eigenvalues are independent (Oct 14)
- Eigenv~~ectors~~alues of a Hermitian matrix are real (Oct 14)
- Factor a matrix A into LU by hand (Homework 1)
- Operations for interval arithmetic (Homework 1)

No computers or calculators will be allowed. Attendance is mandatory to take the test unless you have made arrangements with the DRC or have other special considerations. Please bring your student identification to the exam.

Hausholder reflector... $H = I - 2vv^T$ where v is a unit vector.

$$H^T = (I - 2vv^T)^T = I^T - (2vv^T)^T$$

$$= I^T - 2\,v^{TT}v^T = I - 2vv^T$$

Thus $H^T = H$ and H is symmetric...

$$H^T H = (I - 2vv^T)(I - 2vv^T) = I - 2vv^T - 2vv^T + 4v(v^Tv)v^T$$

$$= I - 4vv^T + 4vv^T = I$$

Interval arithmetic

$$[1,2] + [3,4] = [4,6]$$

What about when there is rounding...

Suppose the arithmetic is good to 4 decimal significant digits. Find the interval corresponding to

$$[3.001,\ 4.002] + [12.04,\ 15.08] = [15.041,\ 19.082]$$

$$= [15.04,\ 19.09]$$

Proof of the quadratic convergence of Newton's method.

Newtons method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Error: $e_n = x_n - \alpha$ where α is the root $f(\alpha)=0$.

Suppose f has a cont. second derivative, $f(\alpha)=0$, $f'(\alpha)\neq 0$. Show that Newton's method converges quadratically provided x_0 is close enough to α at the start.

By Taylor's theorem

$$0 = f(\alpha) = f(x_n - e_n) = f(x_n) - e_n f'(x_n) + \frac{e_n^2}{2} f''(\xi_n)$$

where ξ_n is between x_n and α.

Thus, dividing by $f'(x_n)$ yields

$$0 = \frac{f(x_n)}{f'(x_n)} - e_n + \frac{e_n^2}{2}\frac{f''(\xi_n)}{f'(x_n)}$$

substitute..

Remember quadratic convergence means

$|e_{n+1}| \leq \text{Const } |e_n|^2$

$$e_{n+1} \overset{\text{def of } e_{n+1}}{=} x_{n+1} - \alpha \overset{\text{Newton's method}}{=} x_n - \frac{f(x_n)}{f'(x_n)} - \alpha = e_n - \frac{f(x_n)}{f'(x_n)}$$

$$\overset{\text{by Taylor}}{=} \cancel{e_n} - \left(\cancel{e_n} - \frac{e_n^2}{2}\frac{f''(\xi_n)}{f'(x_n)}\right) = \frac{f''(\xi_n)}{2f'(x_n)} e_n^2$$

The only thing left is to bound by a constant...

Since $f'(\alpha)\neq 0$ there is a δ-neighborhood of α such that $f'(x)\neq 0$ for all $|x-\alpha|\leq\delta$.

define $B = \min\{|f'(x)| : |x-\alpha|\leq\delta\} > 0$

For the same δ define

$$A = \max\{|f''(x)| : |x-\alpha| \le \delta\} < \infty$$

Therefore if $|x_n - \alpha| \le \delta$ and $|\xi_n - \alpha| \le \delta$ then

$$|e_{n+1}| \le \frac{A}{2B}|e_n|^2$$

Need to make sure that $|e_{n+1}| < |e_n|$ so that if $|x_0 - \alpha| \le \delta$ then also $|x_1 - \alpha| \le \delta$ and so forth...

To do this note that

$$|e_{n+1}| \le \boxed{\underbrace{\frac{A}{2B}|e_n|}}\,|e_n|$$

need this less than 1 so the estimate can propagate forward...

Need

$$\frac{A}{2B}|e_0| < 1$$

So if x_0 is close enough to α such that

$$|x_0 - \alpha| < \min\left(\delta, \frac{2B}{A}\right)$$

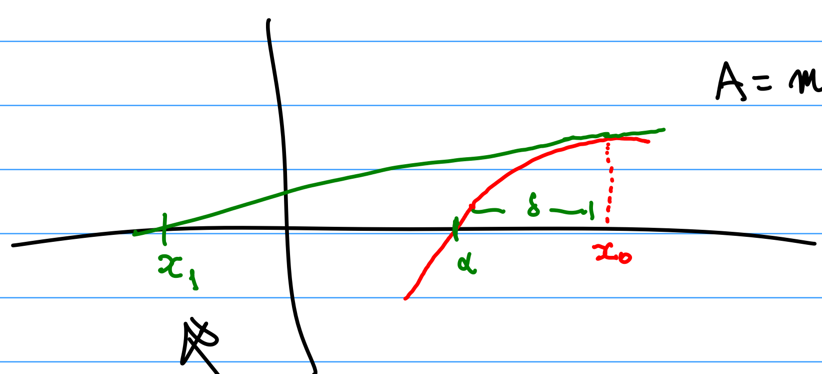
then the Newton's method converges quadratically...

done...

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$B = \min\{|f'(x)| : |x-\alpha| \le \delta\}$$

$$A = \max\{|f''(x)| : |x-\alpha| \le \delta\}$$



$|x_1 - \alpha|$ is bigger than δ so the values for A and B are no longer correct for the next iteration

ensure this doesn't happen by requiring $\frac{A}{2B}|e_0| < 1$

Interpret quadratic convergence

The reason quadratic convergence is important is that the number of correct significant digits essentially doubles at each iteration.

We have $|e_{n+1}| \lesssim M|e_n|^2$

Suppose $|e_n| \approx 10^{-k}$ then

$$|e_{n+1}| \lesssim M(10^{-k})^2 = M \cdot 10^{-2k}$$

twice as accurate except for the M

what if M is really big...

Suppose $M = 10^p$

$$|e_{n+1}| \leq 10^{p-2k}$$

if k is very big compared to p then

$$p-2k \approx -2k$$

and k will be very big after the sequence has iterated for some time...