Math/CS 466/666: Programming Project 2

This project explores the polar decomposition of an invertible matrix. Please form your own teams on WebCampus consisting of 3 to 4 students and welcome any student who wishes to join your group so no people are left out. If you prefer to work independently that is also fine.

Each team should present their work in the form of a typed report using clear and properly punctuated English. Pencil and paper calculations may be typed or hand written. Where appropriate include full program listings and output. Every team member should participate in the work and be prepared to independently answer questions concerning the material. One report per team must be submitted through WebCampus as a single pdf file to complete this project.

1a. Let $A \in \mathbf{R}^{n\times n}$ and consider the iteration

$$X_{k+1} = \frac{1}{2}\Big(X_k + \big(X_k^{-1}\big)^T\Big) \qquad \text{where} \qquad X_0 = A.$$

Write a program to perform this iteration and test your program with the input

$$A = \begin{bmatrix} -0.49 & -0.21 & -0.40 & -0.21 \\ 0.36 & 0.10 & 0.29 & -0.04 \\ 0.12 & -0.01 & 0.48 & -0.47 \\ 0.09 & 0.09 & -0.41 & 0.22 \end{bmatrix}.$$

Verify the Frobenius matrix norm $\|X_0 - X_1\|_F \approx 16.37054203598731$.

b. Define $\Delta_k = X_{k+1} - X_k$. If X_k converges then it follows that $\Delta_k \to 0$ as $k \to \infty$. Compute $\|\Delta_k\|_F$ for $k = 0, \ldots, 9$.

c. Suppose X_k is α-order convergent such that $\|\Delta_{k+1}\|_F \approx M\|\Delta_k\|_F^\alpha$. Then

$$\log\|\Delta_{k+1}\|_F \approx \log M + \alpha \log\|\Delta_k\|_F$$

would show $\log\|\Delta_{k+1}\|_F$ is a linear function of $\log\|\Delta_k\|_F$. Plot the points

$$(\log\|\Delta_k\|_F, \log\|\Delta_{k+1}\|_F) \qquad \text{for} \qquad k = 1, \ldots, 8.$$

Do the points fall on a line? Find the slope between the last two points by computing

$$\alpha \approx \frac{\log\|\Delta_9\|_F - \log\|\Delta_8\|_F}{\log\|\Delta_8\|_F - \log\|\Delta_7\|_F}.$$

d. Let W be the limit of X_k as $k \to \infty$. Numerically check whether W is an orthogonal matrix by computing $X_k^T X_k$ for $k = 8, 9, 10$. What are your conclusions?

e. Define $P = W^{-1}A$ so that $A = WP$. This is called the polar decomposition of A. Use the built-in Julia function `eigvals` to find the eigenvalues of P and A. Are the eigenvalues of A positive? What about the eigenvalues of P?

f. [Extra Credit and for Math 666] Look up the polar decomposition online or in a book on linear algebra and then work Exercise 7.8 from our textbook.