Matrix Norms that come from vector norms

Different vector norms: $x \in \mathbb{R}^n$

Usual Euclidean norm

$$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{x \cdot x}$$

Other norms: for $p \in [1, \infty)$ define

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

properties of a norm

① $\|x+y\| \leq \|x\| + \|y\|$

② $\|\alpha x\| = |\alpha| \, \|x\|$

③ $\|x\| = 0$ if and only if $x = 0$

One more

$$\|x\|_\infty = \max \{ |x_i| : i = 1, \dots, n \}.$$

all of these norms lead to matrix norms

$A \in \mathbb{R}^{m \times n}$ $\qquad Ax \in \mathbb{R}^m \qquad x \in \mathbb{R}^n$

Then

$$\|A\|_p = \max \{ \|Ax\|_p : \|x\|_p = 1 \}$$

matrix norm $\qquad$ vector norm $\qquad$ vector norm

Three norms of particular interest

$p=1$, $p=2$ and $p=\infty$ $\qquad \sqrt[p]{?} = (?)^{1/p}$

interesting because comes from a dot product.

interesting because easy to compute

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \|x\|_1 = \sqrt[1]{\sum_{i=1}^{3} |x_i|^1} = \sum_{i=1}^{3} |x_i| = 6$$

What is $\|A\|_1$? Is it easy to compute?

$A \in \mathbb{R}^{m \times n}$

$$\|A\|_1 = \max\{ \|Ax\|_1 : \|x\|_1 = 1 \}.$$

$$\|Ax\|_1 = \sum_{i=1}^{m} |(Ax)_i| = \sum_{i=1}^{m} \left| \sum_{k=1}^{n} A_{ik} x_k \right|$$

$$(Ax)_i = \sum_{k=1}^{n} A_{ik} x_k$$

triangle inequality

$$\|Ax\|_1 \le \sum_{i=1}^{m} \sum_{k=1}^{n} |A_{ik} x_k| = \overbrace{\sum_{k=1}^{n} \left(\sum_{i=1}^{m} |A_{ik}| \right) |x_k|}^{\text{sum of products...}}$$

depends on S_k

$$S_k = \sum_{i=1}^{m} |A_{ik}|$$

$$= \sum_{k=1}^{n} S_k |x_k|$$

$$\|Ax\|_1 \leq \overbrace{\sum_{k=1}^{n} s_k |x_k|}^{\text{sum of products}}$$

important step...

$$\leq \max\{s_k : k=1,\dots,n\} \sum_{k=1}^{n} |x_k|$$

$$\boxed{\begin{array}{l} \text{Example...} \\ \quad s_1 \cdot |x_1| + s_2 \cdot |x_2| + s_3|x_3| + s_4|x_4| \\ = 2\cdot 3 + 4\cdot 5 + 1\cdot 2 + 3\cdot 8 \\ \leq \max\{2,4,1,3\}(3+5+2+8) \\ = 4(3+5+2+8) \end{array}}$$

$$\|A\|_1 = \max\{\|Ax\|_1 : \|x\|_1 = 1\}.$$

$$\leq \cancel{\max}\{\max\{s_k : k=1,\dots,n\}\underbrace{\cancel{\sum_{k=1}^{n}|x_k|}}_{\text{is actually } \|x\|_1} : \cancel{\|x\|_1 = 1}\}$$

cancel the x's so don't need to take max of x's anymore

$$= \max\{s_k : k=1,\dots,n\}$$

Therefore

$$\|A\|_1 \leq \max\left\{\sum_{i=1}^{m} |A_{ik}| : k=1,\dots,n\right\}$$

Claim this upperbound is sharp, in which case $\|A\|_1 = \max\left\{\sum_{i=1}^{m} |A_{ik}| : k=1,\dots,n\right\}$

Justification of the claim:

Find an unit vector x such that

$$\|Ax\|_1 = \max\left\{ \sum_{i=1}^{m} |A_{ik}| : k=1,\dots,n \right\}$$

Note there is a k_{max} such that (definition of maximum)

$$\sum_{i=1}^{m} |A_{ik_{max}}| = \max\left\{ \sum_{i=1}^{m} |A_{ik}| : k=1,\dots,n \right\}$$

Let $x = e_{k_{max}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← k_{max} position...

$$(Ae_{k_{max}})_i = A_{i,k_{max}}$$

$k_{max} = 2$

$$\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

ith index of this vector

Thus, $x = e_{k_{max}}$

$$\|Ax\|_1 = \sum_{i=1}^{m} |(Ax)_i| = \sum_{i=1}^{m} |A_{i,k_{max}}|$$

Conclusion

$$\|A\|_1 = \max\left\{ \sum_{i=1}^{m} |A_{ik}| : k = 1, \dots, n \right\}$$

rows

Example

$$\left\| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right\|_1 = 18$$

$$\begin{matrix} \| & \| & \| \\ 12 & 15 & 18 \end{matrix}$$

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julia> A=[1 2 3; 4 5 6; 7 8 9]
3×3 Matrix{Int64}:
 1  2  3
 4  5  6
 7  8  9

julia> using LinearAlgebra

julia> opnorm(A,1)
18.0

julia> opnorm(A,2)
16.84810335261421

julia> opnorm(A,Inf)
24.0

julia> norm(A)
16.881943016134134
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← $\|A\|_1$

← $\|A\|_2$

← $\|A\|_\infty$

← $\|A\|_F$

The Frobenius or Engineering norm

$$\begin{matrix} 6 = \\ 15 = \\ 24 = \end{matrix} \left\| \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right\|_\infty = 24$$

Note $\|A\|_\infty = \|A^T\|_1$