...|ε| is small. Thus, an approximation of the con...
...iven by

$$C = \frac{\text{forward error}}{\text{backward error}} = \frac{|(x+\varepsilon)-x|}{|f(x+\varepsilon)-f(x)|}$$

Solving $Ax = b$ and the condition number...

$$\text{(relative)} \quad C = \frac{\text{forward relative error}}{\text{backward relative error}}$$

① Got some way of approximating the solution...

x_* — approximation of the solution...

② Suppose x — exact solution

③ forward relative error $= \dfrac{\|x - x_*\|_p}{\|x\|_p}$ ← vector norms...

④ Backward relative error $= \dfrac{\|b - b_*\|_p}{\|b\|_p}$

where $\boxed{b_* = Ax_*}$ ← this is a different problem to which x_* is the exact solution.

$x_* = A^{-1}b_*$

$\boxed{\text{original problem } b = Ax}$ ← x exact soln to the original problem.

$x = A^{-1}b$

everything I do involves only properties of norms... so it doesn't matter what p is as long as I don't change it halfway through

⑤ Estimate the condition number... (upper bound)

$$C = \frac{\text{forward relative error}}{\text{backward relative error}} = \frac{\|x - x_*\|}{\|x\|} \Big/ \frac{\|b - b_*\|}{\|b\|}$$

$$= \frac{\|x - x_*\|}{\|b - b_*\|} \cdot \frac{\|b\|}{\|x\|} = \frac{\|A^{-1}b - A^{-1}b_*\|}{\|b - b_*\|} \cdot \frac{\|Ax\|}{\|x\|}$$

factor the A^{-1} out

I want to use the inequality $\|Ax\| \le \|A\|\,\|x\|$

$$C = \frac{\|A^{-1}(b-b_*)\|}{\|b-b_*\|} \cdot \frac{\|Ax\|}{\|x\|} \le \frac{\|A^{-1}\|\,\cancel{\|b-b_*\|}}{\cancel{\|b-b_*\|}} \cdot \frac{\|A\|\,\cancel{\|x\|}}{\cancel{\|x\|}} = \|A\|\,\|A^{-1}\|$$

Therefore the condition number of the problem find x such that $Ax=b$ is

$$\boxed{\mathrm{cond}(A) = \|A\|\|A^{-1}\|}$$

What's it good for?

Suppose you find an approximation x_* to the problem $Ax=b$...

plug it in $b_* = Ax_*$ and compute $\dfrac{\|b-b_*\|}{\|b\|} = \dfrac{\|r\|}{\|b\|}$ residual error...

Then

$$\frac{\|x-x_*\|}{\|x\|} \le \mathrm{cond}(A) \cdot \frac{\|b-b_*\|}{\|b\|}$$

How to express the inverse of $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

in terms of the determinant:

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$$

$$\frac{1}{\det(A)}\begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} \alpha\delta - \beta\gamma & \cancel{\beta\delta - \beta\delta}^{\,0} \\ \cancel{-\alpha\gamma + \alpha\gamma}^{\,0} & -\beta\gamma + \alpha\delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If $\mathrm{cond}(A) \approx 10^k$ then its impossible to distinguish two different approximations to x from each other any better than $15-k$ significant digits...

Since we're using double precision arithmetic, the relative error in $\dfrac{\|b-b_*\|}{\|b\|}$ is at best 10^{-15}, can't compute this any more accurately.

If $\mathrm{Cond}(A) = 10^5$, for example, then

$$\frac{\|x - x_*\|}{\|x\|} \lesssim 10^5 \cdot 10^{-15} = 10^{-10}$$

means that x_* is knowable to 10 sig. digits....
based on the ability to compute the backwords error...

that is, based on the ability to numerically check the answer by plugging it in to the original problem.

Details about how to turn in the upcoming homework next week will be posted on the website soon...