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> restart;
> # Determine all two-stage IRK methods of order greater or equal 3
> with(LinearAlgebra):
> m:=4;
                                m:= 4
> # Since order 3 then can check only for autonomous equation
  f:=(t,y)->g(y);
                                f:= (t, y) → g(y)
> A:=Matrix([[a11,a12],[a21,a22]]);
                                A :=  $\begin{bmatrix} a11 & a12 \\ a21 & a22 \end{bmatrix}$ 
> b:=Vector([b1,b2]);
                                b :=  $\begin{bmatrix} b1 \\ b2 \end{bmatrix}$ 
> c:=Vector([a11+a12,a21+a22]);
                                c :=  $\begin{bmatrix} a11 + a12 \\ a21 + a22 \end{bmatrix}$ 
> n:=Dimension(b);
                                n:= 2
> onesvector:=Vector(n,1);
  fxi:=Vector(n):
  for j from 1 to n do
    fxi[j]:=f(t+c[j]*h,xi[j]);
  od:
  fxi;
  xirhs:=y(t)*onesvector+h*Multiply(A,fxi);
                                onesvector :=  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 
                                 $\begin{bmatrix} g(\xi_1) \\ g(\xi_2) \end{bmatrix}$ 
                                xirhs :=  $\begin{bmatrix} y(t) + h (a11 g(\xi_1) + a12 g(\xi_2)) \\ y(t) + h (a21 g(\xi_1) + a22 g(\xi_2)) \end{bmatrix}$ 

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> T:=y(t+h)-y(t)-h*Multiply(Transpose(b), fxi);
for j from 1 to m-1
do
    T:=subs(xi=xirhs,T);
od:

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$$T := y(t + h) - y(t) - h (g(\xi_1) b1 + g(\xi_2) b2)$$

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> T;

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$$\begin{aligned}
 & y(t + h) - y(t) - h (g(y(t) + h (a11 g(y(t) \\
 & + h (a11 g(y(t) + h (a11 g(\xi_1) + a12 g(\xi_2))) + a12 g(y(t) + h (a21 g(\xi_1) + a22 g(\xi_2)))))) + \\
 & a12 g(y(t) \\
 & + h (a21 g(y(t) + h (a11 g(\xi_1) + a12 g(\xi_2))) + a22 g(y(t) + h (a21 g(\xi_1) + a22 g(\xi_2)))))) \\
 & b1 + g(y(t) + h (a21 g(y(t) \\
 & + h (a11 g(y(t) + h (a11 g(\xi_1) + a12 g(\xi_2))) + a12 g(y(t) + h (a21 g(\xi_1) + a22 g(\xi_2)))))) + \\
 & a22 g(y(t) \\
 & + h (a21 g(y(t) + h (a11 g(\xi_1) + a12 g(\xi_2))) + a22 g(y(t) + h (a21 g(\xi_1) + a22 g(\xi_2)))))) \\
 & b2)
 \end{aligned}$$

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> S:=series(T,h,m);

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$$\begin{aligned}
 S := & (-g(y(t)) b1 - g(y(t)) b2 + D(y)(t) h \\
 & + \left(-D(g)(y(t)) g(y(t)) (a21 + a22) b2 - D(g)(y(t)) g(y(t)) (a11 + a12) b1 + \frac{1}{2} D^{(2)}(y)(t) \right) \\
 & h^2 + \left(\right. \\
 & - \left(a22 D(g)(y(t))^2 g(y(t)) a21 + a22^2 D(g)(y(t))^2 g(y(t)) + a21 D(g)(y(t))^2 g(y(t)) a11 \right. \\
 & + a12 D(g)(y(t))^2 g(y(t)) a21 + \frac{1}{2} D^{(2)}(g)(y(t)) a21^2 g(y(t))^2 \\
 & + D^{(2)}(g)(y(t)) a21 g(y(t))^2 a22 + \frac{1}{2} D^{(2)}(g)(y(t)) a22^2 g(y(t))^2 \left. \right) b2 - \left(a12 D(g)(y(t))^2 g(y(t)) \right. \\
 & a21 + a12 D(g)(y(t))^2 g(y(t)) a22 + a11^2 D(g)(y(t))^2 g(y(t)) + a11 D(g)(y(t))^2 g(y(t)) a12 \\
 & + \frac{1}{2} D^{(2)}(g)(y(t)) a11^2 g(y(t))^2 + D^{(2)}(g)(y(t)) a11 g(y(t))^2 a12 \\
 & \left. + \frac{1}{2} D^{(2)}(g)(y(t)) a12^2 g(y(t))^2 \right) b1 + \frac{1}{6} D^{(3)}(y)(t) \left. \right) h^3 + O(h^4)
 \end{aligned}$$

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> eq[1]:=D(t->y(t))(t)=f(t,y(t));

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$$eq_1 := D(y)(t) = g(y(t))$$

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> for j from 1 to m-2
do

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    eq[j+1]:=simplify(subs(seq(eq[i],i=1..j),D(unapply(eq[j],t))(t)));

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od;

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$$eq_2 := D^{(2)}(y)(t) = D(g)(y(t)) g(y(t))$$

$$eq_3 := D^{(3)}(y)(t) = D^{(2)}(g)(y(t)) g(y(t))^2 + D(g)(y(t))^2 g(y(t))$$

> T:=simplify(subs(seq(eq[i],i=1..m-1),S));

$$T := (-g(y(t)) b1 - g(y(t)) b2 + g(y(t))) h + \left(-D(g)(y(t)) g(y(t)) b2 a21 \right. \\ - D(g)(y(t)) g(y(t)) b2 a22 - D(g)(y(t)) g(y(t)) b1 a11 - D(g)(y(t)) g(y(t)) b1 a12 \\ \left. + \frac{1}{2} D(g)(y(t)) g(y(t)) \right) h^2 + \left(-b2 a22 D(g)(y(t))^2 g(y(t)) a21 - b2 a22^2 D(g)(y(t))^2 g(y(t)) \right. \\ - b2 a21 D(g)(y(t))^2 g(y(t)) a11 - b2 a12 D(g)(y(t))^2 g(y(t)) a21 \\ - \frac{1}{2} b2 D^{(2)}(g)(y(t)) a21^2 g(y(t))^2 - b2 D^{(2)}(g)(y(t)) a21 g(y(t))^2 a22 \\ - \frac{1}{2} b2 D^{(2)}(g)(y(t)) a22^2 g(y(t))^2 - b1 a12 D(g)(y(t))^2 g(y(t)) a21 \\ - b1 a12 D(g)(y(t))^2 g(y(t)) a22 - b1 a11^2 D(g)(y(t))^2 g(y(t)) \\ - b1 a11 D(g)(y(t))^2 g(y(t)) a12 - \frac{1}{2} b1 D^{(2)}(g)(y(t)) a11^2 g(y(t))^2 \\ - b1 D^{(2)}(g)(y(t)) a11 g(y(t))^2 a12 - \frac{1}{2} b1 D^{(2)}(g)(y(t)) a12^2 g(y(t))^2 \\ \left. + \frac{1}{6} D^{(2)}(g)(y(t)) g(y(t))^2 + \frac{1}{6} D(g)(y(t))^2 g(y(t)) \right) h^3 + O(h^4)$$

> T1:=coeff(T,h,1);

eq1:=simplify(T1/g(y(t)))=0;

$$T1 := -g(y(t)) b1 - g(y(t)) b2 + g(y(t))$$

$$eq1 := -b1 - b2 + 1 = 0$$

> T2:=coeff(T,h,2);

eq2:=simplify(T2/D(g)(y(t))/g(y(t)))=0;

$$T2 := -D(g)(y(t)) g(y(t)) b2 a21 - D(g)(y(t)) g(y(t)) b2 a22 - D(g)(y(t)) g(y(t)) b1 a11 \\ - D(g)(y(t)) g(y(t)) b1 a12 + \frac{1}{2} D(g)(y(t)) g(y(t))$$

$$eq2 := -b2 a21 - b2 a22 - b1 a11 - b1 a12 + \frac{1}{2} = 0$$

> T3:=coeff(T,h,3);

$$T3 := -b2 a22 D(g)(y(t))^2 g(y(t)) a21 - b2 a22^2 D(g)(y(t))^2 g(y(t)) \\ - b2 a21 D(g)(y(t))^2 g(y(t)) a11 - b2 a12 D(g)(y(t))^2 g(y(t)) a21 \\ - \frac{1}{2} b2 D^{(2)}(g)(y(t)) a21^2 g(y(t))^2 - b2 D^{(2)}(g)(y(t)) a21 g(y(t))^2 a22$$

$$\begin{aligned}
& -\frac{1}{2} b_2 D^{(2)}(g)(y(t)) a_{22}^2 g(y(t))^2 - b_1 a_{12} D(g)(y(t))^2 g(y(t)) a_{21} \\
& - b_1 a_{12} D(g)(y(t))^2 g(y(t)) a_{22} - b_1 a_{11}^2 D(g)(y(t))^2 g(y(t)) \\
& - b_1 a_{11} D(g)(y(t))^2 g(y(t)) a_{12} - \frac{1}{2} b_1 D^{(2)}(g)(y(t)) a_{11}^2 g(y(t))^2 \\
& - b_1 D^{(2)}(g)(y(t)) a_{11} g(y(t))^2 a_{12} - \frac{1}{2} b_1 D^{(2)}(g)(y(t)) a_{12}^2 g(y(t))^2 \\
& + \frac{1}{6} D^{(2)}(g)(y(t)) g(y(t))^2 + \frac{1}{6} D(g)(y(t))^2 g(y(t))
\end{aligned}$$

> **T3a:=coeff(T3,g(y(t)),1);**
eq3:=simplify(T3a/D(g)(y(t))^2)=0;

$$\begin{aligned}
T3a := & -b_2 a_{22} D(g)(y(t))^2 a_{21} - b_2 a_{22}^2 D(g)(y(t))^2 - b_2 a_{21} D(g)(y(t))^2 a_{11} \\
& - b_2 a_{12} D(g)(y(t))^2 a_{21} - b_1 a_{12} D(g)(y(t))^2 a_{21} - b_1 a_{12} D(g)(y(t))^2 a_{22} \\
& - b_1 a_{11}^2 D(g)(y(t))^2 - b_1 a_{11} D(g)(y(t))^2 a_{12} + \frac{1}{6} D(g)(y(t))^2
\end{aligned}$$

$$\begin{aligned}
eq3 := & -b_2 a_{22} a_{21} - b_2 a_{22}^2 - b_2 a_{21} a_{11} - b_2 a_{12} a_{21} - b_1 a_{12} a_{21} - b_1 a_{12} a_{22} \\
& - b_1 a_{11}^2 - b_1 a_{11} a_{12} + \frac{1}{6} = 0
\end{aligned}$$

> **T3b:=coeff(T3,g(y(t)),2);**
eq4:=simplify(2*T3b/D(D(g))(y(t)))=0;

$$\begin{aligned}
T3b := & -\frac{1}{2} b_2 D^{(2)}(g)(y(t)) a_{21}^2 - b_2 D^{(2)}(g)(y(t)) a_{21} a_{22} - \frac{1}{2} b_2 D^{(2)}(g)(y(t)) a_{22}^2 \\
& - \frac{1}{2} b_1 D^{(2)}(g)(y(t)) a_{11}^2 - b_1 D^{(2)}(g)(y(t)) a_{11} a_{12} - \frac{1}{2} b_1 D^{(2)}(g)(y(t)) a_{12}^2 \\
& + \frac{1}{6} D^{(2)}(g)(y(t))
\end{aligned}$$

$$eq4 := -b_2 a_{21}^2 - 2 b_2 a_{22} a_{21} - b_2 a_{22}^2 - b_1 a_{11}^2 - 2 b_1 a_{11} a_{12} - b_1 a_{12}^2 + \frac{1}{3} = 0$$

> **solve({eq1,eq2,eq3,eq4},{a11,a12,a21,a22,b1,b2});**

$$\left\{ \begin{aligned}
b_2 &= \frac{3(1 + 8 a_{11} a_{12} - 4 a_{12} + 4 a_{11}^2 + 4 a_{12}^2 - 4 a_{11})}{4(3 a_{12}^2 + 3 a_{11}^2 - 3 a_{12} - 3 a_{11} + 6 a_{11} a_{12} + 1)}, \\
a_{21} &= -\frac{2 a_{11} - 1 + a_{12}}{3(1 + 8 a_{11} a_{12} - 4 a_{12} + 4 a_{11}^2 + 4 a_{12}^2 - 4 a_{11})}, \\
a_{22} &= \frac{6 a_{11}^2 - 5 a_{11} + 12 a_{11} a_{12} + 1 + 6 a_{12}^2 - 6 a_{12}}{3(2 a_{11} + 2 a_{12} - 1)^2},
\end{aligned} \right.$$

$$\left. b1 = \frac{1}{4(3 a12^2 + 3 a11^2 - 3 a12 - 3 a11 + 6 a11 a12 + 1)}, a11 = a11, a12 = a12 \right\}$$