The Heat Equation

Consider the heat equation on the domain $[0,1]$ given by

$$\begin{cases} u_t = b u_{xx} \\ u(x,0) = 0 \\ u(0,t) = \sin(t) \\ u(1,t) = 0 \end{cases}$$

where $b = 0.01$.

1a. Write a program that uses the explicit finite difference method

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = b \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

to calculate approximate solutions for the heat equation above.

b. Approximate $u(x,t)$ at $t = 2\pi$ using $\Delta x = 1/20$ and $\Delta t = 2\pi/100$.

c. Draw graphs of your approximations to the solutions $u(x,t)$ at $t = 2\pi$ as functions of $x$ using $\Delta x = 1/10, 1/20, 1/30, 1/40, 1/50$ holding $\Delta t = 2\pi/100$ fixed. Explain your results in terms of Fourier stability analysis.

d. Write a program that uses the $\theta$ method method

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \theta b \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2} + (1-\theta) b \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

to calculate approximate solutions for the heat equation above.

e. Verify your program gives the same answers as part b when $\theta = 0$.

f. The Crank-Nicolson method may be obtained by setting $\theta = 1/2$. Draw graphs of your approximations to the solution $u(x,t)$ at $t = 2\pi$ using $\Delta x = 1/20, 1/30, 1/50, 1/100$ holding $\Delta t = 2\pi/100$ fixed. Explain your results. Repeat holding $\Delta t = 2\pi/1000$ fixed.

g. Carefully choose $\theta$, $\Delta x$ and $\Delta t$ to compute the most accurate approximation that you can of $u(x,t)$ when $t = 2\pi$.

h. [Extra Credit and Math/CS 667] Taking your answer from part g to be the exact solution verify the order of convergence numerically in $\Delta t$ and $\Delta x$ for the Crank-Nicolson method.