Math/CS 467/667 Programming Project 3

Kuramoto–Sivashinsky Equation

1. Consider the pattern formation equation

$$u_t + uu_x + \nu u_{xx} + \mu u_{xxxx} = 0 \qquad \text{with} \qquad u(0,x) = u_0(x)$$

on the domain $[-1,1]$ with periodic boundary conditions. Approximate u and uu_x using discrete Fourier series as

$$u(x,t) \approx \sum_{n=-N/2+1}^{N/2} y_j(t) e^{\pi i n x} \quad \text{and} \quad (uu_x)(x,t) \approx \sum_{n=-N/2+1}^{N/2} B_n\big(y(t)\big) e^{\pi i n x}$$

where $y = (y_0, \ldots, y_{N/2}, y_{-N/2+1}, \ldots, y_{-1})$ to obtain the system of ordinary differential equations

$$\frac{dy_n}{dt} + B_n(y) - \nu\pi^2 n^2 y_n + \mu\pi^4 n^4 y_n = 0.$$

Note that B_n depends on t through y and may be computed using the subroutine developed in class for the viscous Burger equations. Write a program to integrate y_n on the interval $[0,T]$ using the split Euler scheme

$$y_{n,j+1} = \big(y_{n,j} - hB_n(y_{\cdot,j})\big)\exp(\nu\pi^2 n^2 h - \mu\pi^4 n^4 h)$$

where $y_{\cdot,j} = (y_{0,j}, \ldots, y_{N/2,j}, y_{-N/2+1,j}, \ldots, y_{-1,j})$ and $y_{n,j} \approx y_n(t_j)$ with $t_j = jh$.

2. Set $u_0(x) = \cos(\pi x) + \sin(3\pi x)$, $\mu = 0.00001$, $\nu = 0.01$, $N = 128$ and $h = T/J$ where $T = 1$ and $J = 16384$. Verify that $u(0,T) \approx 0.32$. Draw a plot of $u(x,T)$ versus x.

3. For convenience define $\alpha_n = \nu\pi^2 n^2 h - \mu\pi^4 n^4 h$ and modify your code to use the split RK2 scheme given by

$$\begin{aligned} k_{1,n} &= -hB_n(y_{\cdot,j}) \\ k_{2,n} &= -he^{-\alpha_n} B_n(p) \qquad \text{where} \qquad p_n = (y_{n,j} + k_{1,n})e^{\alpha_n} \\ y_{n,j+1} &= (y_{n,j} + (k_{1,n} + k_{2,n})/2)e^{\alpha_n}. \end{aligned}$$

Let U^h be the approximation of $u(T)$ using the split RK2 method with step size h. Graph $\log \|U^h - U^{h/2}\|$ versus $\log h$ where $h = 2^{-j}$ for $j = 6, \ldots, 16$ and

$$\|U^h - U^{h/2}\| = \sqrt{\frac{2}{N} \sum_{\ell=-N/2+1}^{N/2} \left| U^h\Big(\frac{2\ell}{N}\Big) - U^{h/2}\Big(\frac{2\ell}{N}\Big)\right|^2}$$

to verify the order of convergence for the split RK2 method numerically. What happens if you take $N = 256$?

4. [Extra Credit] Repeat the previous question for the split RK4 method. Approximate the value of $u(0,T)$ with as much precision as possible by increasing J and N.