Math/CS 467/667 Quiz 1 Version A

1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be an $n+1$ times continuously differentiable function. Prove Taylor's theorem with integral from of the remainder, or in other words, that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + R_n$$

where

$$R_n = \int_x^{x+h} \frac{(x+h-s)^n}{n!} f^{(n+1)}(s)ds.$$

2. Let p_i for $i = 0, \ldots, n$ be a family of orthogonal polynomials such that

$$p_i \text{ has degree } i \qquad \text{and} \qquad \int_{-1}^{1} p_i(x)p_j(x)dx = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Consider the Gaussian quadrature method given by

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=0}^{n-1} w_k f(x_k) \tag{1}$$

where x_k are the n distinct roots such that $p_n(x_k) = 0$ for $k = 0, \ldots, n-1$ and the weights w_k have been chosen such that

$$\int_{-1}^{1} x^j dx = \sum_{k=0}^{n-1} w_k x_k^j \qquad \text{for} \qquad j = 0, \ldots, n-1.$$

Prove the approximation (1) is exact when f is any polynomial of degree $2n - 1$.

3. Answer one of the following two questions:

(i) Show the trapezoid rule

$$\int_a^b f(x)dx \approx \frac{f(a)+f(b)}{2}(b-a)$$

is exact for $f(x) = mx + c$ along any interval $x \in [a, b]$.

(ii) State the weighted mean-value theorem for integrals and then use this theorem to show that R_n as defined in question 1 satisfies

$$R_n = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c) \qquad \text{for some } c \text{ between } x \text{ and } x + h.$$

You may assume $h > 0$ for convenience.

4. Answer one of the following two questions:

(i) Derive α, β and x_0 such that the quadrature rule

$$\int_0^3 f(x)dx \approx \alpha f(x_0) + \beta f(2)$$

holds exactly for polynomials of degree less than or equal 2.

(ii) Suppose f and g are integrable and that $\big|f(x) - g(x)\big| < \epsilon$ for $x \in [a, b]$. Prove

$$\Big|\int_a^b f(x)dx - \int_a^b g(x)dx\Big| \le \epsilon(b - a).$$

5. [Extra Credit] Provide a sequence of twice-differentiable functions

$$f_k\colon [0,1] \to \mathbf{R} \qquad \text{and a twice-differentiable function} \qquad f\colon [0,1] \to \mathbf{R}$$

such that as $k \to \infty$ the following limits hold:

$$\max_{x\in[0,1]} \left| f_k(x) - f(x) \right| \to 0, \qquad \max_{x\in[0,1]} \left| f_k'(x) - f'(x) \right| \to 1$$

and $\max_{x\in[0,1]} \left| f_k''(x) - f''(x) \right| \to \infty$.