Math/CS 467/667: Lecture 1

Taylor's theorem is named after the English mathematician Brook Taylor who originally stated the formula without considering convergence or an estimate of the remainder. Joseph-Louis Lagrange turned Taylor's theorem into a theorem by providing an explicit expression for the error which he used used in his study of mechanics.

Brook Taylor on the left; Joseph-Louis Lagrange on the right.

It is possible to derive Taylor's theorem with an error term given in terms of an integral using the fundamental theorem of calculus and integration by parts. Lagrange's estimate on the error then follows using the weighted mean-value theorem for integrals.

Taylor's Theorem with Integral form of the Remainder. *Let $f : \mathbf{R} \to \mathbf{R}$ be an $n + 1$ times differentiable function. Then*

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + R_n$$

where

$$R_n = \int_x^{x+h} \frac{(x + h - s)^n}{n!} f^{(n+1)}(s)ds.$$

Proof. By the Fundamental Theorem of Calculus

$$f(x + h) - f(x) = \int_x^{x+h} f'(s)ds.$$

Integrating by parts where

$$u = f'(s), \quad dv = ds, \quad du = f''(s)ds \quad \text{and} \quad v = -(x + h - s)$$

yields

$$f(x + h) - f(x) = -(x + h - s)f'(s)\Big|_x^{x+h} + \int_x^{x+h} (x + h - s)f''(s)ds$$

$$= hf'(x) + \int_x^{x+h} (x + h - s)f''(s)ds.$$

Integrating again by parts where

$$u = f''(s), \quad dv = (x + h - s)ds, \quad du = f^{(3)}(s)ds \quad \text{and} \quad v = -\tfrac{1}{2}(x + h - s)^2$$

yields

$$f(x + h) - f(x) = hf'(x) - \frac{(x + h - s)^2}{2} f''(s)\Big|_x^{x+h} + \int_x^{x+h} \frac{(x + h - s)^2}{2} f^{(3)}(s)ds$$

$$= hf'(x) + \frac{h^2}{2} f''(x) + \int_x^{x+h} \frac{(x + h - s)^2}{2} f^{(3)}(s)ds.$$

Repeated integration by parts finishes the proof.   ////

A weight function on the interval $[a, b]$ is a function w such that

$$w(s) \geq 0 \quad \text{for} \quad s \in [a, b] \quad \text{and} \quad 0 < \int_a^b w(s)ds < \infty.$$

A simple argument leads to

The Weighted Mean-value Theorem for Integrals. *Let f be continuous on the interval $[a, b]$ and w a weight function. Then, there is $c \in (a, b)$ such that*

$$\int_a^b f(s)w(s)ds = f(c) \int_a^b w(s)ds.$$

Proof. Note that the average value of f is between the maximum and minimum values of f on the interval $[a, b]$. Since f is continuous, it must pass through all values between its minimum and maximum and, in particular must pass through its average value. Now, take c to be the point where $f(c)$ is equal to its average value.   ////

Now use the weighted mean-value theorem to transform the integral form of the remainder to Lagrange's expression. We remark that this approach requires the derivative $f^{(n+1)}$ to be continuous whereas Lagrange's original theorem was based on the mean-value theorem for derivatives and only required the weaker hypothesis that $f^{(n+1)}$ exists.

Taylor's Theorem with Lagrange form of the Remainder. *Let $f : \mathbf{R} \to \mathbf{R}$ be an $n + 1$ times differentiable function such that $f^{(n+1)}$ is continuous. Then*

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + R_n$$

where

$$R_n = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(c) \quad \text{for some } c \text{ between } x \text{ and } x + h.$$

Proof. Case $h > 0$. Define $w(s) = (x + h - s)^n/n!$ and note that w is a weight function on the interval $[x, x + h]$. If $h < 0$ the argument is similar.   ////