## Math/CS 467/667: Lecture 3

Given a quadrature rule

$$\texttt{quad}(f) = \sum_{k=0}^{n-1} w_k f(x_k) \qquad \text{such that} \qquad \int_{-1}^{1} f(x)dx \approx \texttt{quad}(f) \tag{1}$$

is exact when $f$ is a polynomial of degree less than or equal $N$, consider the approximation

$$\int_a^b f(x)dx \approx \texttt{comp}(f,a,b,m) \qquad \text{where} \qquad \texttt{comp}(f,a,b,m) = \sum_{i=0}^{m-1} \texttt{quad}(g_i)$$

is the composite quadrature formula on $[a,b]$ over $m$ subintervals given by

$$g_j(x) = \frac{h}{2} f\Big(\frac{xh}{2} + a + hj + \frac{h}{2}\Big) \qquad \text{and} \qquad h = \frac{b-a}{m}.$$

Before beginning our analysis of the composite quadrature formula $\texttt{comp}(f,a,b,m)$, we first prove a lemma that results from the monotonicity of $\texttt{quad}(f)$ when the weights $w_k$ are positive but also holds, in general, when they have mixed signs. Note that there are examples of naturally occurring Newton–Cotes quadrature formulas for which some of the weights $w_k$ turn out to be negative. In the case when `quad` is given by Gaussian quadrature we have $N = 2n-1$ and the weights are positive.

**Lemma 2.** *There is a constant $c \ge 2$ depending only on $n$ and the $w_k$'s such that*

$$|f| \le M \qquad \textit{implies} \qquad |\texttt{quad}(f)| \le cM$$

**Proof.** By the triangle inequality

$$|\texttt{quad}(f)| \le \sum_{k=0}^{n-1} |w_k| \big|f(x_k)\big| \le cM \qquad \text{where} \qquad c = \sum_{k=0}^{n-1} |w_k|.$$

In the case the $w_k \ge 0$ for all $k$ we further have that

$$c = \sum_{k=0}^{n-1} |w_k| = \sum_{k=0}^{n-1} w_k = \texttt{quad}(1) = \int_{-1}^{1} 1 \cdot dx = 2.$$

Therefore, take $c = 2$ when all the weights are non-negative and note that $c > 2$ when some of the weights are negative. This finishes the proof of the lemma. ////

Note under the assumption the weights $w_k$ are non-negative the approximation `quad` is, in fact, monotone as can be seen as follows: Suppose $f(x) \le g(x)$ for all $x$, then

$$\texttt{quad}(f) = \sum_{k=0}^{n-1} w_k f(x_k) \le \sum_{k=0}^{n-1} w_k g(x_k) = \texttt{quad}(f).$$

This, in particular, implies $|\texttt{quad(}f\texttt{)}| \le \texttt{quad(}|f|\texttt{)}$, which means the approximation of the area under the absolute value of a function is always larger than the absolute value of the approximation of its integral.

We characterize now the error in the composite quadrature formula by proving

**Theorem 3.** *If $f$ has $N+1$ continuous derivatives on the interval $[a,b]$ then the error*

$$E_m = \Big| \int_a^b f(x)dx - \texttt{comp(}f,a,b,m\texttt{)} \Big| = \mathcal{O}(h^{N+1}) \qquad \textit{as} \qquad h \to 0.$$

**Proof.** Let $t_j = a + hj$ and note that

$$\int_a^b f(x)dx = \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} f(t)dt.$$

For each of the integrals over the intervals $[t_j, t_{j+1}]$ of length $h$ appearing on the right hand side make the change of variables

$$t = \frac{-t_j(x-1)}{2} + \frac{t_{j+1}(x+1)}{2} = \frac{xh}{2} + a + hj + \frac{h}{2} \qquad \text{and} \qquad dt = \frac{h}{2}dx$$

to obtain

$$\int_{t_j}^{t_{j+1}} f(t)dt = \frac{h}{2}\int_{-1}^{1} f\Big(\frac{xh}{2} + a + hj + \frac{h}{2}\Big)dx = \int_{-1}^{1} g_j(x)dx.$$

We now use the fact that `quad` is exact for polynomials of degree less or equal $N$ to obtain bounds on the error. By the triangle inequality

$$E_m \le \sum_{j=0}^{m-1} \Big| \int_{-1}^{1} g_j(x)dx - \texttt{quad(}g_j\texttt{)} \Big|. \tag{4}$$

Since $f$ has $N+1$ continuous derivatives and the maximum of the continuous function $f^{(N+1)}(x)$ is guaranteed to exist on the closed interval $[a,b]$, then we may define

$$M = \max\big\{ |f^{N+1}(x)| : x \in [a,b] \big\}.$$

Upon noting that $g_j$ also has $N+1$ continuous derivatives, it follows from Taylor's theorem that $g_j(x) = T_j(x) + R_j(x)$ where $T_j$ is the Taylor polynomial of degree $N$ expanded about $x = 0$ and $R_j$ is the remainder given by

$$R_j(x) = \frac{x^{N+1}}{(N+1)!} g_j^{(N+1)}(\xi_j) \qquad \text{for some } \xi_j \text{ between 0 and } x.$$

Since $x \in [-1,1]$ then $\xi_j \in [-1,1]$. By the chain rule we obtain

$$\begin{aligned}
|g_j^{(N+1)}(\xi_j)| &= \frac{h}{2}\Big| \Big(\frac{d}{dx}\Big)^{N+1} f\Big(\frac{xh}{2} + a + hj + \frac{h}{2}\Big)\Big|_{x=\xi_j} \\
&= \Big(\frac{h}{2}\Big)^{N+2} \Big| f^{(N+1)}\Big(\frac{\xi_j h}{2} + a + hj + \frac{h}{2}\Big)\Big| \\
&\le \Big(\frac{h}{2}\Big)^{N+2} \max\big\{ |f^{(N+1)}(t) : t \in [t_j, t_{j+1}] \big\} \le \Big(\frac{h}{2}\Big)^{N+2} M.
\end{aligned}$$

Consequently,

$$|R_j(x)| \le \frac{|x|^{N+1}}{(N+1)!}\Big(\frac{h}{2}\Big)^{N+2} M \le h^{N+2}B \qquad \text{where} \qquad B = \frac{1}{(N+1)!}\cdot\frac{M}{2^{N+2}}.$$

Plugging the Taylor polynomial and remainder into (4) and using the fact that `quad` is exact for polynomials of degree less than or equal $N$ we obtain

$$E_m \le \sum_{j=0}^{m-1}\Big|\int_{-1}^{1} R_j(x)dx\Big| + \sum_{j=0}^{m-1}\big|\texttt{quad(}R_j\texttt{)}\big|.$$

At this point we use the monotonicity of the integral—the fact that the area under the absolute value of a curve is greater than the original area—to estimate

$$\Big|\int_{-1}^{1} R_j(x)dx\Big| \le \int_{-1}^{1}\big|R_j(x)\big|dx \le \int_{-1}^{1} h^{N+2}B = 2h^{N+2}B.$$

Combining the above estimate with the Lemma 2 applied to $|\texttt{quad(}R_j\texttt{)}|$ yields

$$\begin{aligned} E_m &\le \sum_{i=0}^{m-1}(2+c)h^{N+2}B = (2+c)mh^{N+2}B \\ &= (2+c)B(b-a)h^{N+1} = \mathcal{O}(h^{N+1}) \qquad \text{as} \qquad h \to 0. \end{aligned}$$

This finishes the proof of the theorem. ////

We remark in the case of Gaussian quadrature where $N = 2n-1$ that the results of Theorem 3 may be simplified to obtain

$$E_m \le 4B(b-a)\Big(\frac{b-a}{m}\Big)^{2n}.$$

In applications one typically chooses $n$ fixed and then increases $m$ until the desired error goals are met. It is, of course, possible to increase $n$ as well. However, in doing so, one must remember that $B$ also depends on $n$ through $M$.

If $f$ is an analytic function such that its Taylor series converges on a closed disk in the complex plane of radius $r$ at every point $x \in [a,b]$, this means for complex $\omega$ that

$$\max\Big\{\frac{r^{2n}}{(2n)!}|f^{(2n)}(\omega)| : |\omega - x| \le r\Big\} \le \max\big\{\,|f(\omega)| : |\omega - x| = r\,\big\}$$

for all $n \ge 0$. It follows that $(2+c)B(2r)^{2n}(b-a)$ is bounded, say by $A$, and consequently it holds for every $h < 2r$ that

$$E_m \le A\Big(\frac{h}{2r}\Big)^{2n} \to 0 \qquad \text{exponentially as} \qquad n \to \infty.$$

Thus, provided $h$ is small enough, it is also possible—though less common—to meet any error bounds with exponential efficiency by taking $n$ sufficiently large.