Math 467/667: Programming Project 2

Your work should be presented in the form of a typed report using clear and properly punctuated English. Pencil and paper calculations may be typed or hand written. Where appropriate include full program listings and output. Please work individually on this project.

1. Let $y(x)$ be the exact solution to the differential equation

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

Let $x_n = x_0 + nh$ and define the approximation $y_n \approx y(x_n)$ using the Shu–Osher three-stage Runge–Kutta numerical method

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + h, y_n + hk_1) \\ k_3 &= f(x_n + h/2, y_n + h(k_1 + k_2)/4) \\ y_{n+1} &= y_n + h(k_1 + k_2 + 4k_3)/6. \end{aligned}$$

(i) Determine the truncation error by substituting the exact solution into the numerical method and making a series expansion around $h = 0$. Namely, define

$$\begin{aligned} \kappa_1 &= f(x_n, y(x_n)) \\ \kappa_2 &= f(x_n + h, y(x_n) + h\kappa_1) \\ \kappa_3 &= f(x_n + h/2, y(x_n) + h(\kappa_1 + \kappa_2)/4) \end{aligned}$$

and then find m such that

$$\tau_n = y(x_{n+1}) - y(x_n) - h(\kappa_1 + \kappa_2 + 4\kappa_3)/6 = \mathcal{O}(h^m) \qquad \text{as} \qquad h \to 0.$$

If you use a computer algebra system to do this (recommended), please include both the input and output for the calculation.

(ii) Determine the linear stability domain of the Shu–Osher method. Namely, substitute $f(x, y) = Ay$ into the method, set $h = 1$ and exactly solve the resulting difference equation for a solution of the form $y_n = \rho^n$. Write $A = a + ib$ and plot what values of $a + ib$ in the complex plane lead to values of ρ such that $|\rho| < 1$.

(iii) Consider the ordinary differential equation

$$y' = y^2 \sin 2x, \qquad y(0) = -1$$

on the interval $[0, 4]$. Find the exact solution.

(iv) Use the Shu–Osher method to approximate the solution to the differential equation solved in part (iii) by setting $h = 4/N$ with $N = 32$. Plot both the exact and the approximate solutions on the same graph. Include program source code, numerical output to be plotted as well as the actual plot. Comment on the accuracy of the approximation.

(v) Define the error

$$E(h) = \max\left\{ |y_n - y(x_n)| : n = 0, 1, \ldots, N \right\}.$$

If $E(h) \approx Kh^p$ for some K and some p we say that the method is of order p. Numerically determine the order of the Shu–Osher method by approximating the differential equation in part (iii) using $N = 2^j$ for $j = 5, 6, \ldots, 16$. Plot the corresponding values of $E(h)$ versus h using log-log coordinates. Include source code, the numerical values of $E(h)$ as well as the resulting plot in your report.

(vi) Theoretically the value of p found in part (v) should satisfy $p \approx m - 1$ where m is determined by the truncation error in part (i). Use this fact to verify that your computer program is working as expected and discuss how well the theory holds.

(vii) [Extra Credit and for Math 667] Note that the exact solution found in part (iii) is bounded for all time $t > 0$. Explore the stability of the numerical approximation by choosing larger and larger values h to find the maximum value h_* such that the resulting numerical approximation y_n remains bounded for all n. Is it possible to relate the linear stability domain determined in part (ii) to the value of h_* by linearizing the non-linear equation in part (iii) about the exact solution?