Math/CS 467/667: Lecture 2

In 1814 Gauss [2] developed a method for approximating definite integrals using an optimal quadrature formula of the form

$$\int_a^b f(x)dx \approx \sum_{k=0}^{n-1} w_k f(x_k)$$

that is exact for polynomials of degree $2n-1$. Since a $2n-1$ degree polynomial is determined by $2n$ coefficients and the values of x_k and w_k for $k=0,1,\ldots,n-1$ represent $2n$ parameters, the existence of such a formula seems reasonable. To overcome the difficulty of directly solving the resulting $2n$ non-linear equations for $2n$ unknowns, Gauss employed the family of orthogonal polynomials that were developed by Legendre [4] as solutions to differential equations. Extensions of these techniques were provided by Christoffel [1] in 1877 who obtained the existence and uniqueness of optimal quadratures for a general class of weighted integrals. Additional information may be found in Gautschi [3].

Adrien-Marie Legendre on the left; Carl Friedrich Gauss center; Elwin Bruno Christoffel on the right.

Orthogonal Polynomials

Rather than following Legendre who describes the orthogonal polynomials p_n of degree n on the interval $[-1,1]$ as solutions to the differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0,$$

we instead use the Gram–Schmidt orthogonalization process.

Consider the inner product and norm on the space of integrable functions defined by

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx \qquad \text{and} \qquad \|f\| = \sqrt{(f,f)}.$$

The orthogonal polynomials

$$\{\, p_k : k = 0, 1, \ldots n \,\}$$

may be obtained using the Gram–Schmidt orthogonalization procedure with respect to the above inner product and norm starting with the standard polynomial basis

$$\{\, x^k : k = 0, 1, \dots n \,\}.$$

In particular, the orthogonal polynomials are given by

$$\begin{aligned}
v_0 &= 1 & p_0 &= \frac{v_0}{\|v_0\|} \\
v_1 &= x - (x, p_0)p_0 & p_1 &= \frac{v_1}{\|v_1\|} \\
v_2 &= x^2 - (x^2, p_0)p_0 - (x^2, p_1)p_1 & p_2 &= \frac{v_2}{\|v_2\|} \\
&\vdots & &\vdots \\
v_n &= x^n - \sum_{k=0}^{n-1} (x^n, p_k)p_k & p_n &= \frac{v_n}{\|v_n\|}.
\end{aligned}$$

Construction of Gauss Quadrature

The points x_k and the weights w_k used in the approximation

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{k=0}^{n-1} w_k f(x_k)$$

that we shall call Gauss quadrature are given as follows. Let x_k for $k = 0, 1, \dots, n-1$ be the n distinct roots to the orthogonal polynomial p_n of degree n. Thus $p_n(x_k) = 0$ for $k = 0, 1, \dots, n-1$. We remark without proof that the x_k's are real and moreover that $x_k \in [-1, 1]$. Now, consider the system of n linear equations given by

$$\int_{-1}^{1} x^j\, dx = \sum_{k=0}^{n-1} w_k x_k^j \qquad \text{for} \qquad j = 0, 1, \dots, n-1$$

in the n unknowns w_k where $k = 0, 1, \dots n-1$. Since the x_j's are distinct this system is non-singular. Therefore, there exists a unique solution for the w_k's. This specifies the x_k's and w_k's in the Gauss quadrature formula.

Accuracy of Gauss Quadrature

In this section we prove Gauss quadrature is exact for polynomials of degree $2n - 1$.

Proof. Let p be a polynomial of degree $2n - 1$. Since the p_n has degree n, the division algorithm implies there exist polynomials r and q of degree $n - 1$ such that

$$p(x) = q(x)p_n(x) + r(x).$$

Claim that

$$\int_{-1}^{1} r(x)dx = \sum_{k=0}^{n-1} w_k r(x_k).$$

Write

$$r(x) = \sum_{j=0}^{n-1} a_j x^j.$$

Then by the choice of w_k's we have

$$\begin{aligned}\int_{-1}^{1} r(x)dx &= \int_{-1}^{1} \sum_{j=0}^{n-1} a_j x^j dx = \sum_{j=0}^{n-1} a_j \int_{-1}^{1} x^j dx \\ &= \sum_{j=0}^{n-1} a_j \sum_{k=0}^{n-1} w_k x_k^j = \sum_{k=0}^{n-1} w_k \sum_{j=0}^{n-1} a_j x_k^j = \sum_{k=0}^{n-1} w_k r(x_k).\end{aligned}$$

Since p_n is orthogonal to all polynomials of degree $n-1$ or less and $p_n(x_k) = 0$, then

$$\begin{aligned}\int_{-1}^{1} p(x)dx &= \int_{-1}^{1} \big(q(x)p_n(x) + r(x)\big)dx = (q, p_n) + \int_{-1}^{1} r(x)dx \\ &= \int_{-1}^{1} r(x)dx = \sum_{k=0}^{n-1} w_k r(x_k) = \sum_{k=0}^{n-1} w_k \big(q(x_k)\cdot 0 + r(x_k)\big) \\ &= \sum_{k=0}^{n-1} w_k \big(q(x_k)p_n(x_k) + r(x_k)\big) = \sum_{k=0}^{n-1} w_k p(x_k).\end{aligned}$$

This finishes the proof. ////

References

1. E. B. Christoffel, Sur une Class Particulière de Fonctions Entiéres et de Fractions Continues, *Ann. Mat. Pura Appl.*, Vol. 8, 1877, pp. 1–10.
2. C. F. Gauss, Methodus Nova Integralium Valores per Approximationem Inveniendi, *Comment. Soc. Reginae Sci. Gottingensis Rencentiores*, Vol. 3, 1815, pp. 163–196.
3. W. Gautschi, Construction of Gauss-Christoffel Quadrature Formulas, *Math. Comp.*, Vol 22, 1968, pp. 251–270.
4. A. M. Legendre, Recherches sur la Figure des Planètes, *Histoire de l'Académie Royale des Sciences*, 1784, pp. 1–370.