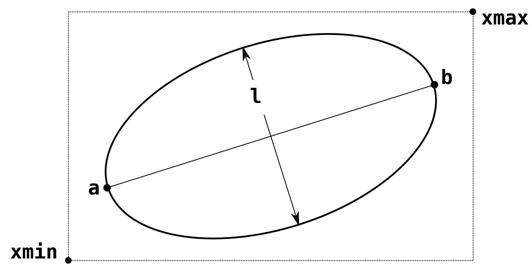
Math/CS 467/667 Programming Assignment 2

1. The two-dimensional Poisson equation is the elliptic partial differential equation $\Delta u = f$ for $x \in \Omega$ with boundary conditions $u(x) = \psi(x)$ for $x \in \partial\Omega$. To verify the performance of the finite-difference method start with a differentiable function that will serve as the exact solution. For this assignment let

$$u_{\text{ex}}(x) = 2x_1^4 \cos 3x_2 + e^{2x_2} \sin 2x_1.$$

Now, define $\psi(x) = u_{\text{ex}}(x)$ on $\partial\Omega$ and find f by computing Δu_{ex}. What is the value of $f(x)$?

2. Let Ω be an ellipse contained in a bounding box between $x_{\min}$ and $x_{\max}$ of the form



where a and b are the endpoints of the major axis and ℓ is the length of the minor axis. For

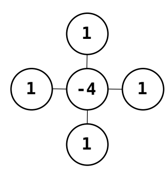
$$a = (1.5, 2), \qquad b = (6.5, 3), \qquad \ell = 2.6, \qquad x_{\min} = (1,1) \qquad \text{and} \qquad x_{\max} = (7,4)$$

find a function $\texttt{domain}(x)$ such that $\Omega = \{\, x \in \mathbf{R}^2 : \texttt{domain}(x) > 1 \,\}$. Note there are many choices that will work equally well. One idea is to use

$$\texttt{domain}(x) = 2 - (x-c)^T R^T S^2 R(x-c)$$

where c is the center of the ellipse, S is a diagonal matrix with the values $2/|a-b|$ and $2/\ell$ on the diagonal and R is a suitable rotation.

3. Subdivide the bounding box horizontally using $\texttt{mres} = 100$ grid points and vertically with the same spacing. This is how the program `poisson.jl` from class creates the grid. Then use the 5-point stencil



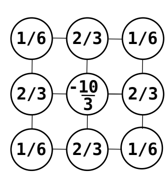
to create an approximate solution u_{ap} to Poisson's equation. Define

$$\|u\|^2 = \sum_{x \in \mathcal{J}} \big|u(x)\big|^2 h^2 \qquad \text{where} \qquad h = \frac{6}{\texttt{mres} - 1}, \qquad \mathcal{J} = \{\, x_{ij} : x_{ij} \in \Omega \,\}$$

and x_{ij} are the grid points. Find the error $\|u_{\text{ap}} - u_{\text{ex}}\|$ in your approximation and check that it is approximately 6.5213971543769285. What is the relative error $\|u_{\text{ap}} - u_{\text{ex}}\| / \|u_{\text{ex}}\|$?

4. Let $E_n = \|u_{\text{ap}} - u_{\text{ex}}\|$ be the error when $\texttt{mres} = n$. Verify the order of convergence of the 5-point stencil is $\mathcal{O}(h^2)$ by checking that $E_n/E_{2n} \approx 4$ for $n = 100$, 200, 400 and 800. Although the boundary is only resolved with $\mathcal{O}(h)$, why in this case does that not affect the convergence of the solution?

5. [Extra Credit and Math 667] Modify your program to use the the 9-point stencil



Compute E_n/E_{2n} for $n = 100$, 200, 400 and 800. What is the apparent order of the method? Is this consistent with the theoretical result you expected? Replace f by $g = f + \frac{1}{12}h^2\Delta f$. Now what happens?