Math 488 Example Problem

$$u_t = c^2 u_{xx} + xt$$

$$u(0,t) = \sin t$$

$$2u(\pi,t) + 3u_x(\pi,t) = e^{-x}$$

$$u(x,0) = g(x)$$

First write $u(x,t) = w(x,t) + \sum v_n(x,t)$ where $w(x,t)$ is some function that satisfies the boundary conditions.

In this case $w(x,t) = m(t)x + b(t)$ where

$$w(0,t) = b(t) = \sin t$$

and

$$2w(\pi,t) + 3w_x(\pi,t) = 2\big(m(t)\pi + \sin t\big) + 3m(t) = e^{-t}$$

implies

$$(2\pi+3)m(t) = e^{-t} - 2\sin t$$

$$m(t) = \frac{e^{-t} - 2\sin t}{2\pi + 3}.$$

Thus

$$w(x,t) = \frac{e^{-t} - 2\sin t}{2\pi+3}\,x + \sin t$$

Writting $U = \sum u_n(x,t)$. We obtain that

$$w_t + U_t = c^2 w_{xx} + c^2 U_{xx} + xt$$

and so

$$U_t = c^2 U_{xx} + F(x,t)$$

where $F(x,t) = xt + c^2 w_{xx} - w_t$.

In this case $w_{xx}=0$ and

$$w_t = -\frac{e^{-t}+2\cos t}{2\pi+3}x + \cos t$$

So

$$U_t = c^2 U_{xx} + xt + \frac{e^{-t}+2\cos t}{2\pi+3}x - \cos t$$

We now solve this equation subject to the boundary conditions

$$U(0,t)=0$$

$$2U(\pi,t)+3U_x(\pi,t)=e^{-t}$$

and initial condition

$$U(x,0)=g(x)-w(x,0)=g(x)-\frac{x}{2\pi+3}$$

We look first for special solutions to

$$v_t = c^2 v_{xx}$$

$$v(0,t)=0$$

$$2v(\pi,t)+3v_x(\pi,t)=0$$

using separation of variables

$$v(x,t)=X(x)T(t)$$

$$XT' = c^2X''T$$

$$\frac{T'}{c^2T} = \frac{X''}{X} = k$$

We solve the ODE

$$X''=kX$$

$$X(0)=0$$

$$2X(\pi)+3X'(\pi)=0$$

Non-trivial solutions are when $k<0$.

$$X = A\cos\sqrt{|k|}\,x + B\sin\sqrt{|k|}\,x$$

$$X' = -A\sqrt{|k|}\sin\sqrt{|k|}\,x + B\sqrt{|k|}\cos\sqrt{|k|}\,x$$

$$X(0) = A = 0$$

$$2X(\pi) + 3X'(\pi) = 2B\sin\sqrt{|k|}\,\pi + 3B\sqrt{|k|}\cos\sqrt{|k|}\,\pi = 0$$

Since we want $B \neq 0$ then

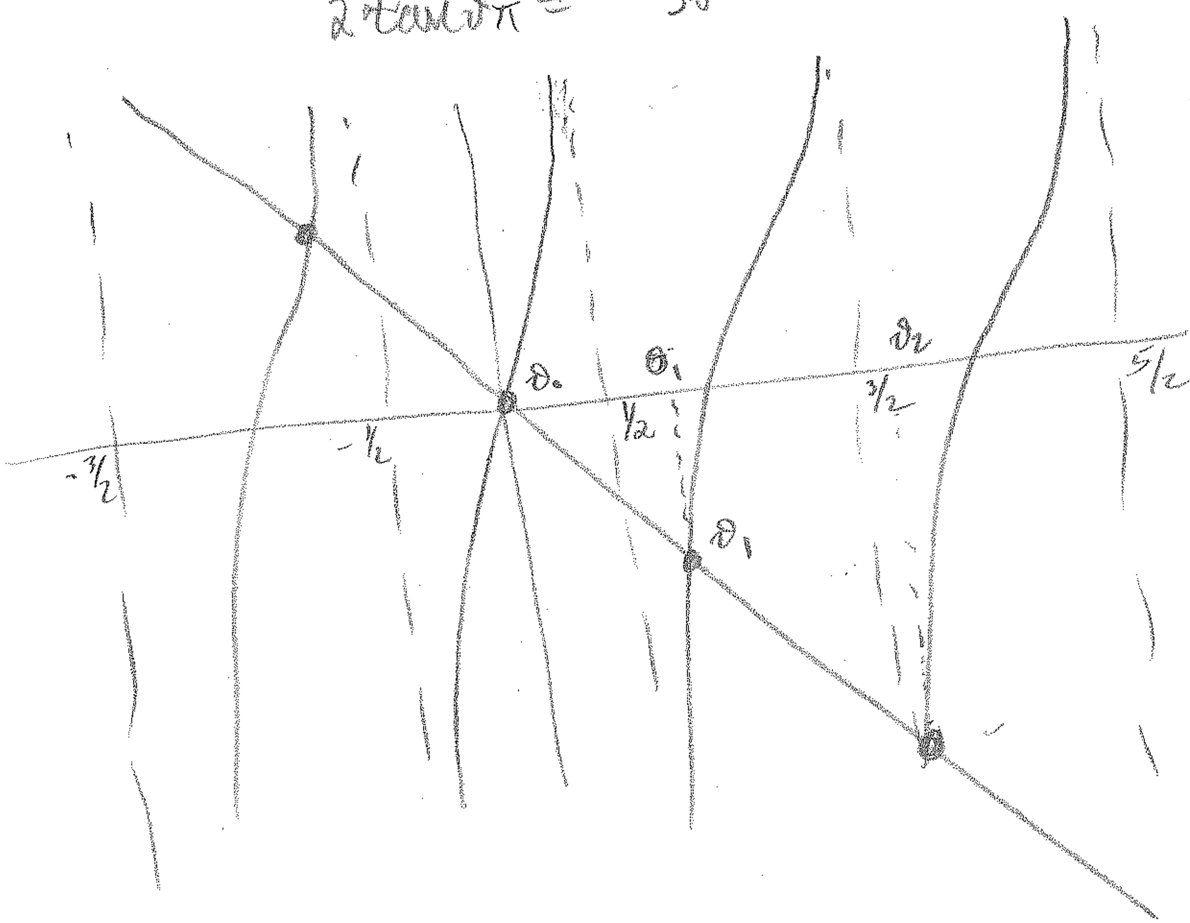
$$2\sin\sqrt{|k|}\,\pi + 3\sqrt{|k|}\cos\sqrt{|k|}\,\pi = 0$$

Let $\vartheta = \sqrt{|k|}$ so that $k = -\vartheta^2$. Then ϑ satisfies the equation

$$2\sin\vartheta\pi + 3\vartheta\cos\vartheta\pi = 0$$

or

$$2\tan\vartheta\pi = -3\vartheta$$



There are infinite number of such solutions. If ϑ is a solution so is $-\vartheta$. We are only interested in the positive solutions.

Let ϑ_n be the n^{th} positive solution ordered from small to large.

Note that $\vartheta_n \in [n-\frac{1}{2}, n+\frac{1}{2}]$.

and that $\vartheta_n - n + \frac{1}{2} \to 0$ as $n \to \infty$.

Claim that $\int_0^\pi \sin\vartheta_n x \sin\vartheta_m x\, dx = 0$ for $m \neq n$.

This can be seen using integration by parts.

$$\int_0^\pi \sin\vartheta_n x \sin\vartheta_m x\, dx = \frac{-1}{\vartheta_m}\sin\vartheta_n x \cos\vartheta_m x\Big|_0^\pi + \frac{\vartheta_n}{\vartheta_m}\int_0^\pi \cos\vartheta_n x \cos\vartheta_m x\, dx$$

$$= \frac{-1}{\vartheta_m}\sin\vartheta_n\pi\cos\vartheta_m\pi + \frac{\vartheta_n}{\vartheta_m^2}\cos\vartheta_n x \sin\vartheta_m x\Big|_0^\pi + \frac{\vartheta_n^2}{\vartheta_m^2}\int_0^\pi \sin\vartheta_n x \sin\vartheta_m x\, dx$$

Thus using that $\vartheta\cos\vartheta\pi = -\frac{2}{3}\sin\vartheta\pi$ we have

$$(\vartheta_m^2 - \vartheta_n^2)\int_0^\pi \sin\vartheta_n x \sin\vartheta_m x\, dx$$

$$= -\vartheta_m \sin\vartheta_n\pi\cos\vartheta_m\pi + \vartheta_n\cos\vartheta_n\pi\sin\vartheta_m\pi$$

$$= \frac{2}{3}\sin\vartheta_n\pi\sin\vartheta_m\pi - \frac{2}{3}\sin\vartheta_n\pi\sin\vartheta_m\pi$$

$$= 0$$

If $n = m \neq 0$ then

$$\int_0^\pi \sin^2 \vartheta_n x\, dx = \int_0^\pi \frac{1-\cos 2\vartheta_n x}{2}\, dx$$

$$= \frac{\pi}{2} - \frac{\sin 2\vartheta_n x}{4\vartheta_n}\Big|_0^\pi = \frac{\pi}{2} - \frac{\sin 2\vartheta_n \pi}{4\vartheta_n} > 0$$

Therefore we look for a solution

$$U(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \vartheta_n x$$

That solves.

$$U_t = c^2 U_{xx} + xt + \frac{e^t + 2\cos t}{2\pi + 3} x - \cos t$$

$$U(0,t) = 0$$

$$2U(\pi,t) + 3U_x(x,t) = e^{x}$$

$$U(x,0) = g(x) - \frac{x}{2\pi+3}$$

This is similar to the problem worked last Friday

$$\text{let } F(x,t) = xt + \frac{e^t + 2\cos t}{2\pi+3} x - \cos t$$

Then

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin \vartheta_n x$$

where $F_n(t)$ are given by multiplying this equation by $\sin \vartheta_m x$ and integrating.

$$\int_0^{\pi} F(x,t)\sin\vartheta_m x\,dx = \sum_{n=1}^{\infty} F_n(t)\int_0^{\pi}\sin\vartheta_n x\sin\vartheta_m x\,dx$$

$$= F_m(t)\left(\frac{\pi}{2} - \frac{\sin 2\vartheta_m\pi}{4\vartheta_m}\right) = F_m(t)\,\frac{2\pi\vartheta_m - \sin 2\vartheta_m\pi}{4\vartheta_m}$$

So

$$F_m(t) = \frac{4\vartheta_m}{2\pi\vartheta_m - \sin 2\vartheta_m\pi}\int_0^{\pi} F(x,t)\sin\vartheta_m x\,dx$$

$$= \frac{4\vartheta_m}{2\pi\vartheta_m - \sin 2\vartheta_m\pi}\int_0^{\pi}\left(xt + \frac{e^t + 2\cos t}{2\pi+3}x - \cos t\right)\sin\vartheta_m x\,dx$$

$$= \frac{4\vartheta_m}{2\pi\vartheta_m - \sin 2\vartheta_m\pi}\left(\frac{-1}{\vartheta_m}\left(xt + \frac{e^t+2\cos t}{2\pi+3}x\right)\cos\vartheta_m x\Big|_0^{\pi}\right.$$

$$\left. + \frac{1}{\vartheta_m}\int_0^{\pi}\left(t + \frac{e^t+2\cos t}{2\pi+3}\right)\cos\vartheta_m x\,dx + \frac{\cos t\cos\vartheta_m x}{\vartheta_m}\Big|_0^{\pi}\right)$$

$$= \frac{4\vartheta_m}{2\pi\vartheta_m - \sin 2\vartheta_m\pi}\left(\frac{-\pi}{\vartheta_m}\left(t + \frac{e^t+2\cos t}{2\pi+3}\right)\cos\vartheta_m\pi\right.$$

$$\left. + \frac{1}{\vartheta_m^2}\left(t + \frac{e^t+2\cos t}{2\pi+3}\right)\sin\vartheta_m\pi + \frac{\cos t}{\vartheta_m}\left(\cos\vartheta_m\pi - 1\right)\right)$$

Since

$$\frac{\sin\vartheta_m\pi}{\vartheta_m} = -\frac{3}{2}\cos\vartheta_m\pi$$

$$F_m(t) = \frac{4}{2\pi\vartheta_m - \sin 2\vartheta_m\pi}\left(\left(t + \frac{e^t + 2\cos t}{2\pi+3}\right)(\cos\vartheta_m\pi)\left(-\pi - \frac{3}{2}\right) + \cos t\left(\cos\vartheta_m\pi - 1\right)\right)$$

Similarly write

$$G(x) = g(x) - \frac{x}{2\pi+3}$$

as

$$G(x) = \sum_{n=1}^{\infty} G_n \sin\vartheta_n x$$

where

$$G_m = \frac{4\vartheta_m}{2\pi\vartheta_m - \sin 2\vartheta_m\pi}\int_0^{\pi} G(x)\sin\vartheta_m x\,dx.$$

Now solve the ODEs for the coefficients $a_n(t)$ from the equations

$$a_n'(t) = -c^2\vartheta_n^2 a_n(t) + F_n(t)$$

$$a_n(0) = G_n$$

By integrating factor

$$\frac{d}{dt}a_n(t)e^{c^2\vartheta_n^2 t} = F_n(t)e^{c^2\vartheta_n^2 t}$$

$$a_n(t) = a_n(0)e^{-c^2\vartheta_n^2 t} + \int_0^t F_n(\tau)e^{c^2\vartheta_n^2(\tau - t)}\,d\tau$$

$$= G_n e^{-c^2\vartheta_n^2 t} + \int_0^t F_n(\tau)e^{c^2\vartheta_n^2(\tau - t)}\,d\tau.$$

The complete solution is then

$$u(x,t) = w(x,t) + v(x,t)$$

$$= \frac{e^{-t} - 2\sin t}{2\pi + 3} x + \sin t + \sum_{n=1}^{\infty} a_n(t) \sin \vartheta_n x$$

where ϑ_n are the solutions to

$$2\sin\vartheta\pi + 3\vartheta\cos\vartheta\pi = 0$$

and

$$a_n(t) = G_n e^{-c^2\vartheta_n^2 t} + \int_0^t F_n(\tau) e^{c^2\vartheta_n^2(\tau - t)} d\tau$$

and F_n and G_n are given on page 7.

By Maple

$$F_n(t) = -\frac{2((2t\pi + 3t + e^t)\cos\pi\vartheta_n + 2\cos t}{\vartheta_n(3\cos^2(\vartheta_n\pi) + 2\pi)}$$

and

$$G_n(t) = g_n + \frac{3\pi^2\vartheta_n\cos(\vartheta_n\pi)}{(3\cos^2(\vartheta_n\pi) + 2\pi)(2\pi + 3)}$$

and

$$g_m = \frac{4\vartheta_m}{2\pi\vartheta_m - \sin 2\vartheta_m\pi} \int_0^\pi g(x) \sin\vartheta_m x \, dx.$$

```
> restart;
> nm:=int(sin(theta*x)^2,x=0..Pi);
```

$$nm := \frac{-\cos(\pi\,\theta)\,\sin(\pi\,\theta) + \pi\,\theta}{2\,\theta}$$

```
> F:=x*t+(exp(t)+2*cos(t))/(2*Pi+3)*x-cos(t);
```

$$F := x\,t + \frac{(e^t + 2\cos(t))\,x}{2\,\pi + 3} - \cos(t)$$

```
> Fm:=1/nm*int(F*sin(theta*x),x=0..Pi);
```

$$Fm := \frac{1}{(-\cos(\pi\,\theta)\,\sin(\pi\,\theta) + \pi\,\theta)\,\theta\,(2\,\pi + 3)}\,(2\,(-2\,\theta\cos(t)\,\pi - 3\,\theta\cos(t) + 2\,t\,\pi\sin(\pi\,\theta) + 3\,t\sin(\pi\,\theta) + 2\cos(t)\sin(\pi\,\theta) - 2\,t\,\pi^2\,\theta\cos(\pi\,\theta) - 3\,t\,\pi\,\theta\cos(\pi\,\theta) - e^t\,\theta\cos(\pi\,\theta)\,\pi + e^t\sin(\pi\,\theta) + 3\,\theta\cos(t)\cos(\pi\,\theta)))$$

```
> a2:=simplify(subs(sin(theta*Pi)=-3/2*theta*cos(theta*Pi),Fm));
```

$$a2 := -\frac{2\,(2\,t\,\pi\cos(\pi\,\theta) + 2\cos(t) + 3\,t\cos(\pi\,\theta) + e^t\cos(\pi\,\theta))}{\theta\,(3\cos(\pi\,\theta)^2 + 2\,\pi)}$$

```
> #There are the coefficents Fm
  a3:=collect(a2,cos(Pi*theta));
```

$$a3 := -\frac{2\,((2\,t\,\pi + 3\,t + e^t)\cos(\pi\,\theta) + 2\cos(t))}{\theta\,(3\cos(\pi\,\theta)^2 + 2\,\pi)}$$

```
> Gm:=1/nm*int(-x/(2*Pi+3)*sin(theta*Pi),x=0..Pi);
```

$$Gm := -\frac{\theta\sin(\pi\,\theta)\,\pi^2}{(-\cos(\pi\,\theta)\sin(\pi\,\theta) + \pi\,\theta)\,(2\,\pi + 3)}$$

```
> #These are the coefficents for Gm-gm
  simplify(subs(sin(theta*Pi)=-3/2*theta*cos(theta*Pi),Gm));
```

$$\frac{3\,\theta\cos(\pi\,\theta)\,\pi^2}{(3\cos(\pi\,\theta)^2 + 2\,\pi)\,(2\,\pi + 3)}$$

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>
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