— Key —

Math 488/688 Final Version A

Instructions: Undergraduate students work 5 of the following 7 problems; graduate students work 6 of the following 7 problems.

1. Solve the first order partial differential equation

$$\begin{cases} u_x + 2u_y = \cos y \\ u(x,0) = \cos 3x. \end{cases}$$

$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2 \qquad \frac{du}{dt} = \cos y$

$x = t + C \qquad y = 2t \qquad u = \int \cos 2t \, dt = \frac{1}{2}\sin 2t + D$

$u(t+C, 2t) = \frac{1}{2}\sin 2t + D$

set $t = 0$

$u(C, 0) = \cos 3C = D, \qquad D = \cos 3C$

$u(t+C, 2t) = \frac{1}{2}\sin 2t + \cos 3C$

$x = t + C, \quad y = 2t \qquad t = y/2$

$C = x - t = x - y/2$

$u(x,y) = \frac{1}{2}\sin y + \cos(3x - \frac{3}{2}y)$

2. Solve the first order partial differential equation

$$\begin{cases} y^2 u_x + (x+1)u_y = 0 \\ u(0,y) = y^3. \end{cases}$$

$$\frac{dx}{dt} = y^2 \qquad \frac{dy}{dt} = x+1 \qquad \frac{du}{dt} = 0$$

$$\frac{\left(\frac{dx}{dt}\right)}{\left(\frac{dy}{dt}\right)} = \frac{dx}{dy} = \frac{y^2}{x+1}$$

$$\int (x+1)\,dx = \int y^2\,dy$$

$$\frac{x^2}{2} + x + C = \frac{1}{3}y^3$$

Solve for y: $\quad y = \sqrt[3]{\frac{3x^2}{2} + 3x + 3C}$

Parameterize the characteristic with x and C.

Thus

$$\frac{d}{dx}u\left(x, \sqrt[3]{\frac{3x^2}{2} + 3x + 3C}\right) = \frac{du}{dt}\frac{dt}{dx} = 0 \cdot \frac{1}{\frac{dx}{dt}} = 0$$

So $\quad u\left(x, \sqrt[3]{\frac{3x^2}{2} + 3x + 3C}\right) = D$

Set $x = 0$, $\quad u(0, \sqrt[3]{3C}) = 3C = D \quad D = 3C$

$$u\left(x, \sqrt[3]{\frac{3x^2}{2} + 3x + 3C}\right) = 3C$$

$$y = \sqrt[3]{\frac{3x^2}{2} + 3x + 3C}, \quad C = \frac{1}{3}y^3 - \frac{x^2}{2} - x$$

Therefore

$$u(x,y) = y^3 - \frac{3}{2}x^2 - 3x.$$

3. Solve the second order partial differential equation

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0, y) = 0 \\ u(1, y) = 0 \\ u(x, 0) = 0 \\ u(x, 1) = 2 \sin 7\pi x \end{cases}$$

separation of variables.

$X''Y + XY'' = 0$, $\quad \frac{X''}{X} = -\frac{Y''}{Y} = k$

Solve $\quad X'' = kX$, $X(0) = X(1) = 0$

case $k<0$: $\quad X = A\cos\sqrt{|k|}\, x + B\sin\sqrt{|k|}\, x$

$X(0) = A = 0$, $\quad A = 0$

$X(1) = B\sin\sqrt{|k|} = 0 \quad \sqrt{|k|} = n\pi$, $n = 1, 2, 3, \dots$

Case $k \geq 0$: no nontrivial solutions.

Therefore $\quad k = -n^2\pi^2$ for $n = 1, 2, 3, \dots$

Solve $\quad Y'' = n^2\pi^2 Y$, $Y(0) = 0$

$Y = C\sinh n\pi y + D\cosh n\pi y$

$Y(0) = D = 0$, $\quad D = 0$

We write the solution as

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin n\pi x \sinh n\pi y$$

Satisfy the remaining boundary condition:

$$u(x,1) = \sum_{n=1}^{\infty} B_n \sin n\pi x \sinh n\pi = 2\sin 7\pi x.$$

Therefore $B_n = 0$ if $n \neq 7$. and $B_7 = \frac{2}{\sinh 7\pi}$

The solution is

$$u(x,y) = \frac{2}{\sinh 7\pi} \sin 7\pi x \sinh 7\pi y$$

4. Solve the second order partial differential equation

$$\begin{cases} u_t = 4u_{xx} + e^{-\pi^2 t}\cos \pi x \\ u_x(0,t) = 0 \\ u_x(2,t) = 0 \\ u(x,0) = 0. \end{cases}$$

Separation of variables for the homogeneous equation gives. $XT' = 4X''T$, $\frac{T'}{4T} = \frac{X''}{X} = k$

Solve $X'' = kX$, $X'(0) = X'(2) = 0$

case $k \le 0$, $X = A\cos\sqrt{|k|}x + B\sin\sqrt{|k|}x$

$X' = -A\sqrt{|k|}\sin\sqrt{|k|}x + B\sqrt{|k|}\cos\sqrt{|k|}x$

$X(0) = B\sqrt{|k|} = 0$, $B = 0$

$X(2) = -A\sqrt{|k|}\sin\sqrt{|k|}2 = 0$

so $2\sqrt{|k|} = n\pi$, $n = 0, 1, 2, \ldots$ or $\sqrt{|k|} = \frac{n\pi}{2}$

case $k > 0$, non non-trivial solutions.

Try a series solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\cos\frac{n\pi x}{2}$$

Thus

$$\sum_{n=0}^{\infty} a_n'(t)\cos\frac{n\pi x}{2} = -4\sum_{n=0}^{\infty}\frac{n^2\pi^2}{4}a_n(t)\cos\frac{n\pi x}{2} + e^{-\pi^2 t}\cos\pi x$$

equating coefficients.

$a_n' + n^2\pi^2 a_n = 0$, $a_n(0) = 0$ for $n \ne 2$

$a_2' + 4\pi^2 a_2 = e^{-\pi^2 t}$, $a_2(0) = 0$ for $n = 2$

In the first case $a_n = 0$ for $n \ne 2$. In the second

$$(a_2 e^{4\pi^2 t})' = e^{3\pi^2 t}$$

$$a_2(t) = a_2(0)e^{-4\pi^2 t} + \frac{1}{3\pi^2}(e^{-\pi^2 t} - e^{-4\pi^2 t})$$

Therefore

$$u(x,t) = \frac{1}{3\pi^2}(e^{-\pi^2 t} - e^{-4\pi^2 t})\cos\pi x$$

5. Solve the second order partial differential equation

$$\begin{cases} u_{tt} = u_{xx} \\ u(0,t) = 0 \\ u(1,t) = t^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

First write $u = w + v$ where w satisfies the boundary.

$w(x,t) = a(t)x + b(t)$

$w(0,t) = b(t) = 0 \qquad w(1,t) = a(t) = t^2$

$w(x,t) = t^2 x$

Now $w_t = 2tx$, $w_{tt} = 2x$, $w_{xx} = 0$, $w(x,0) = 0$, $w_t(x,0) = 0$

Therefore v satisfies.

$$\begin{cases} v_{tt} = v_{xx} - 2x \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = 0. \\ v_t(x,0) = 0 \end{cases}$$

Seperation of variables for the homogeneous eq. yields.

$XT'' = X''T, \qquad \dfrac{T''}{T} = \dfrac{X''}{X} = k$

Solve $X'' = kX$, $X(0) = X(1) = 0$

$X = A \sin n\pi x$ as in problem 3 earlier.

Now write $-2x = \sum_{n=1}^{\infty} f_n \sin n\pi x$

$$f_n = -4\int_0^1 x \sin n\pi x \, dx = \frac{4}{n\pi}\left(x\cos n\pi x \Big|_0^1 - \int_0^1 \cos n\pi x \, dx \right)$$

$$= \frac{4}{n\pi} \cos n\pi = \frac{4(-1)^n}{n\pi}$$

Now write $v(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin n\pi x$ and plug in to get

$$\sum_{n=1}^{\infty} a_n''(t) \sin n\pi x = \sum_{n=1}^{\infty} -n^2\pi^2 a_n(t) \sin n\pi x + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n\pi} \sin n\pi x$$

which gives the system of ODEs

$$a_n'' + n^2\pi^2 a_n = \frac{4(-1)^n}{n\pi}, \quad a_n(0) = 0, \quad a_n'(0) = 0$$

These can be solved by writing as a particular solution plus a solution to the homogeneous problem.

particular solution

$$p_n'' + n^2\pi^2 p_n = \frac{4(-1)^n}{n\pi}$$

$$p_n = \frac{4(-1)^n}{n^3\pi^3}$$

homogeneous eq

$$h_n'' + n^2\pi^2 h_n = 0 \qquad h_n(0) = -\frac{4(-1)^n}{n^3\pi^3}, \; h_n'(0) = 0$$

$$h_n = A\cos n\pi t + B\sin n\pi t,$$

$$h_n(0) = A = \frac{-4(-1)^n}{n^3\pi^3}$$

$$h_n' = -An\pi\sin n\pi t + Bn\pi\cos n\pi t$$

$$h_n'(0) = Bn\pi = 0, \qquad B = 0$$

Thus

$$a_n = \frac{4(-1)^n}{n^3\pi^3}\left(1 - \cos n\pi t\right)$$

It follows that

$$v(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^3\pi^3}(1-\cos n\pi t)\sin n\pi x$$

and

$$u(x,t) = t^2 x + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^3\pi^3}(1-\cos n\pi t)\sin n\pi x$$

6. Find the Green's function for

$$\begin{cases} u'' + u = f \\ u(0) = 0 \\ u'(5) = 0 \end{cases}$$

Green's function is of the form

$$g(x, x_0) = \begin{cases} C\, u_\ell(x) u_r(x_0) & x \le x_0 \\ C\, u_\ell(x_0) u_r(x) & x \ge x_0 \end{cases}$$

where u_ℓ and u_r satisfy the homogeneous equations

$u_\ell'' + u_\ell = 0, \quad u_\ell(0) = 0,$

$u_\ell = A \cos x + B \sin x$

$u_\ell(0) = A = 0$, take $B = 1$

$u_\ell = \sin x$

$u_r'' + u_r = 0 \quad u_r'(5) = 0$

$u_r = a \cos(x-5) + b \sin(x-5)$

$u_r' = -a \sin(x-5) + b \cos(x-5)$

$u_r'(5) = b = 0$, take $a = 1$

$u_r = \cos(x-5)$

Thus

$$g(x, x_0) = \begin{cases} C \sin x \cos(x_0 - 5) & x \le x_0 \\ C \sin x_0 \cos(x-5) & x \ge x_0 \end{cases}$$

Now solve for C using the jump condition

$$C\left(\cos x_0 \cos(x_0 - 5) - \sin x_0 \sin(x_0 - 5)\right) = 1$$

Since the problem is self adjoint C doesn't depend on x_0.
Taking $x_0 = 0$ yields

$$C \cos 5 = 1 \quad \text{so} \quad C = \frac{1}{\cos 5}$$

Therefore

$$g(x, x_0) = \begin{cases} \dfrac{1}{\cos 5} \sin x \cos(x_0 - 5) & x \le x_0 \\[2ex] \dfrac{1}{\cos 5} \sin x_0 \cos(x-5) & x \ge x_0 \end{cases}$$

7. Find the Green's function for

$$\begin{cases} u'' - 2u' + 5u = f \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

Rewrite as
$$(u'e^{-2x})' + 5e^{-2x}u = fe^{-2x} = F$$
and find the green's function such that
$$u = -\int_0^1 G(x, x_0)\, F(x_0)\, dx_0 \quad \text{first.}$$
Thus
$$G(x, x_0) = \begin{cases} c\, u_l(x) u_r(x_0) & x \le x_0 \\ c\, u_l(x_0) u_r(x) & x \ge x_0 \end{cases}$$
where

$u_l'' - 2u_l' + 5u_l = 0$, $u_l(0) = 0$. $\quad r^2 - 2r + 5 = 0$

$(r-1)^2 + 4 = 0$

$u_l = e^x(A\cos 2x + B\sin 2x)$ $\quad r = 1 \pm 2i$

$u_l(0) = A = 0$ $\quad A = 0$. Take $B = 1$ for simplicity.

$u_l = e^x \sin 2x$.

shift to get $u_r = e^{x-1}\sin 2(x-1)$. Now solve for the jump condition

$$c\, e^{-2x_0}\Big(\big(u_l(x_0) + 2e^{x_0}\cos 2x_0\big)e^{x_0-1}\sin 2(x_0-1)$$
$$- e^{x_0}\sin 2x_0\,\big(u_r(x_0) + 2e^{x_0-1}\cos 2(x_0-1)\big)\Big) = 1$$

Setting $x_0 = 0$ yields
$$c \cdot 1 \cdot \Big[(0+2)e^{-1}\sin(-2) - 0\Big] = 1, \quad c = \frac{-e}{2\sin 2}$$

Now

$$g(x, x_0) = G(x, x_0)e^{-2x_0}$$
$$= \begin{cases} \dfrac{-e}{2\sin 2} e^{-2x_0} e^{x}\sin 2x\, e^{x_0-1}\sin 2(x_0-1) & x \le x_0 \\[2ex] \dfrac{-e}{2\sin 2} e^{-2x_0} e^{x_0}\sin 2x_0\, e^{x-1}\sin 2(x-1) & x \ge x_0 \end{cases}$$