Math 488/688 Quiz 1 Version A

1. Derive either the heat equation or the wave equation from physical principles. There will be partial credit for simply stating the equation and explaining what all the variables and constants mean.

See text

Evans, Blackledge & Yardly
pgs 3 – 5

or

Farlow
pg 28 and pg 123 - 125

Math 488/688 Quiz 1 Version A

2. Solve the first order partial differential equation

$$\begin{cases} 2u_x + 3u_y = xy \\ u(2,y) = \sin y \end{cases}$$

Characteristics:

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 3$$

$$x = 2t \qquad y = 3t + C$$

solve ODE along characteristics

$$\frac{du}{dt} = 2t(3t+C) = 6t^2 + 2Ct$$

$$u = 2t^3 + Ct^2 + D$$

solve initial condition

$$u(2t, 3t+C) = 2t^3 + Ct^2 + D$$

$t = 1$

$$u(2, 3+C) = 2 + C + D = \sin(3+C)$$

$$D = \sin(3+C) - 2 - C$$

Thus

$$u(2t, 3t+C) = 2t^3 + Ct^2 + \sin(3+C) - 2 - C$$

rewriting in terms of x and y yields

$$x = 2t \qquad t = \frac{x}{2} \qquad y = 3t + C \qquad C = y - \frac{3}{2}x$$

Therefore

$$u(x,t) = \frac{1}{4}x^3 + \left(y - \frac{3}{2}x\right)\frac{x^2}{4} + \sin\left(3 + y - \frac{3}{2}x\right) - 2 - y + \frac{3}{2}x$$

Math 488/688 Quiz 1 Version A

3. [Extra Credit] Solve the first order partial differential equation

$$\begin{cases} u_x + (x-y)u_y = yu \\ u(0,y) = 3 \end{cases}$$

characteristics:

$\frac{dx}{dt} = 1$ $\quad \frac{dy}{dt} = (t-y)$ $\quad \frac{dy}{dt} + y = t$

$x = t$ $\quad \frac{d}{dt}(ye^t) = te^t$

$$ye^t = \int te^t dt = te^t - \int e^t dt$$

$$ye^t = (t-1)e^t + C$$

$$y = t - 1 + Ce^{-t}$$

Solve along the characteristics.

$$\frac{du}{dt} = (t - 1 + Ce^{-t})u$$

$$\ln u = \frac{t^2}{2} - t - Ce^{-t} + D$$

$$u = Ae^{\frac{t^2}{2} - t - Ce^{-t}}$$

Solve the initial data

$$u(t, t-1+Ce^{-t}) = Ae^{\frac{t^2}{2} - t - Ce^{-t}} = 3e^{\frac{t^2}{2} - t + C(1-e^{-t})}$$

$t=0,$ $\quad u(0, C-1) = Ae^{-C} = 3 \qquad A = 3e^{C}$

rewrite in terms of x and y

$x = t \quad y = x - 1 + Ce^{-x} \qquad C = (1-x+y)e^{x}$

$$u(x,y) = 3e^{\left[\frac{x^2}{2} - x + (1-x+y)(e^{x}-1)\right]}$$