HW #2 solutions

#1 $u_t = 2u_{xx} + 3x$

$u(0,t) = 0 \quad u(2,t) = 0$

$u(x,0) = x(x-2)$

Solve homogeneous eq. in space first.

$v_t = 2v_{xx}$

$v(0,t) = 0 \quad v(2,t) = 0$.

Separation of vbls.

$XT' = 2X''T$

$$\frac{T'}{2T} = \frac{X''}{X} = k$$

$X'' = kX, \quad X(0) = 0, \quad X(2) = 0$

$k<0 \quad X = A\cos\sqrt{|k|}x + B\sin\sqrt{|k|}x$

$X(0) = A = 0 \quad A = 0$

$X(2) = B\sin\sqrt{|k|}\,2 = 0$

$\sqrt{|k|}\,2 = n\pi \quad n = 1, 2, 3, \ldots$

General solution to

$$u(x,t) = \sum b_n(t)\sin\frac{n\pi x}{2}$$

Write the force as

$$3x = \sum f_n \sin\frac{n\pi x}{2}$$

$$f_n = \frac{2}{2}\int_0^2 3x\sin\frac{n\pi x}{2}\,dx = -\frac{12}{n\pi}(-1)^n$$

Substituting into the original differential equation

$$\sum_{n=1}^{\infty} b_n'(t) \sin \frac{n\pi x}{2} = \lambda \sum_{n=1}^{\infty} b_n(t) \left(-\frac{n^2\pi^2}{4}\right) \sin \frac{n\pi x}{2} - \sum_{n=1}^{\infty} \frac{12}{n\pi} (-1)^n \sin \frac{n\pi x}{2}$$

equating coefficients and integrating

$$b_n'(t) + \frac{n^2\pi^2}{2} b_n(t) = \frac{-12}{n\pi} (-1)^n$$

$$\left(b(t) e^{\frac{n^2\pi^2}{2} t}\right)' = -\frac{12}{n\pi} (-1)^n e^{\frac{n^2\pi^2}{2} t}$$

$$b(t) e^{\frac{n^2\pi^2}{2} t} - b(0) = \int_0^t \frac{-12}{n\pi} (-1)^n e^{\frac{n^2\pi^2}{2} t} dt$$

$$= -\frac{24}{n^3\pi^3} (-1)^n \left(e^{\frac{n^2\pi^2}{2} t} - 1\right)$$

Thus

$$b(t) = b(0) e^{-\frac{n^2\pi^2}{2} t} - \frac{24}{n^3\pi^3} (-1)^n \left(1 - e^{-\frac{n^2\pi^2}{2} t}\right)$$

Now solve for $b(0)$.

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n\pi x}{2}$$

$$u(x,0) = \sum_{n=0}^{\infty} b_n(0) \sin \frac{n\pi x}{2} = x(x-2)$$

$$b_n(0) = \int_0^2 x(x-2) \sin \frac{n\pi x}{2} dx = \frac{16}{n^3\pi^3} \left((-1)^n - 1\right)$$

Therefore

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{16}{n^3\pi^3}\left((-1)^n - 1\right) e^{-\frac{n^2\pi^2}{2} t} - \frac{24}{n^3\pi^3} (-1)^n \left(1 - e^{-\frac{n^2\pi^2}{2} t}\right)\right] \sin \frac{n\pi x}{2}.$$

#2 $u_t = u_{xx} + \pi^2 u$

$u_x(0,t) = 0$

$u_x(1,t) = 0$

$u(x,0) = \cos(\pi x)$

Use separation of variables

$$XT' = X''T + \pi^2 XT = T(X'' + \pi^2 X)$$

$$\frac{T'}{T} = \frac{X'' + \pi^2 X}{X} = k$$

$$X'' + \pi^2 X = kX \quad , \; X'(0) = 0 , \; X'(1) = 0$$

$$X'' = (k - \pi^2)X$$

$k - \pi^2 < 0$ $\quad X = A\cos\sqrt{|k-\pi^2|}\,x + B\sin\sqrt{|k-\pi^2|}\,x$

$$X' = -A\sqrt{\#}\sin\sqrt{\#}\,x + B\sqrt{\#}\cos\sqrt{\#}\,x$$

$$X'(0) = B\sqrt{\#} = 0 \qquad B = 0$$

$$X'(1) = -A\sqrt{\#}\sin\sqrt{\#} = 0$$

so $\sqrt{|k-\pi^2|} = n\pi \qquad n = 1, 2, 3, \ldots$

$$k - \pi^2 = -n^2\pi^2$$

$$k = (1-n^2)\pi^2$$

$k - \pi^2 = 0$ $\quad X = Ax + B$

$$X' = A$$

$$X'(0) = 0 \qquad A = 0$$

$$X'(1) = 0$$

$$X = B$$

which is already included if we take $n = 0$ in the solutions of the form $A\cos n\pi x$.

Solving for T gives

$$T' = (1-n^2)\pi^2 T$$
$$T = e^{(1-n^2)\pi^2 t}$$

The general solutions are

$$u(x,t) = \sum_{n=0}^{\infty} A_n(\cos n\pi x)\, e^{(1-n^2)\pi^2 t}$$

Thus the initial condition yields

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos n\pi x = \cos \pi x$$

and so $A_1 = 1$ and $A_n = 0$ for $n \neq 1$. It follows that

$$u(x,t) = (\cos \pi x)\, e^{(1-1^2)\pi^2 t} = \cos \pi x.$$

#3 $u_{tt} = 4u_{xx}$

$u(0,t)=0$

$u(3,t)=0$

$$u(x,0) = \begin{cases} 2x & \text{if } x\in 1 \\ 3-1 & \text{if } x>1 \end{cases}$$

$u_t(x,0)=0$

separation of variables gives

$$XT'' = 4X''T$$

$$\frac{T''}{4T} = \frac{X''}{X} = k$$

$$X'' = kX \quad , \quad X(0)=0, \quad X(3)=0$$

$k<0$ $\quad X = A\cos\sqrt{|k|}\,x + B\sin\sqrt{|k|}\,x$

$X(0) = A = 0 \qquad A=0$

$X(3) = B\sin\sqrt{|k|}\,3 = 0$

$\sqrt{|k|}\,3 = n\pi \qquad n=1,2,\ldots$

$k = -\left(\frac{n\pi}{3}\right)^2$

$$T'' = -4\left(\frac{n\pi}{3}\right)^2 T \quad , \quad T'(0)=0$$

$$T = F\sin\frac{2n\pi t}{3} + G\cos\frac{2n\pi t}{3}$$

$$T' = \frac{2n\pi}{3}\left(F\cos\frac{2n\pi t}{3} - G\sin\frac{2n\pi t}{3}\right)$$

$$T'(0) = \frac{2n\pi}{3}F = 0 \qquad F=0$$

General solution is

$$u(x,t) = \sum_{n=1}^{\infty} G_n \sin\frac{n\pi x}{3}\cos\frac{2n\pi t}{3}$$

Initial condition

$$u(x,0) = \sum_{n=1}^{\infty} G_n \sin\frac{n\pi x}{3} = \begin{cases} 2x & \text{if } x \le 1 \\ 3-x & \text{if } x > 1 \end{cases}$$

$$\frac{3}{2} G_n = \int_0^1 2x \sin\frac{n\pi x}{3}\,dx + \int_1^3 (3-x)\sin\frac{n\pi x}{3}\,dx = \frac{27}{n^2\pi^2}\sin\frac{n\pi}{3}$$

Therefore $G_n = \frac{18}{n^2\pi^2}\sin\frac{n\pi}{3}$ and

$$u(x,t) = \sum_{n=1}^{\infty} \frac{18}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{3}\cos\frac{2n\pi t}{3}$$

#4 $U_t = 3U_{xx}$

$U_x(0,t)=0$

$U(2,t)=1$

$U(x,0)=\cos \pi x$

First write the solution as $U = w+v$ where w satisfies

$w_x(0,t)=0$

$w(2,t)=1$

Solving for $w(x,t)=a(t)x+b(t)$ yields

$w_x(0,t)=a(t)=0$

$w(2,t)=b(t)=1$

Thus $w(x,t)=1$ and v must solve

$v_t = 3v_{xx}$

$v_x(0,t)=0$

$v(2,t)=0$

$v(x,0)=\cos \pi x - 1$

Separation of variables gives

$$XT' = 3X''T$$

$$\frac{T'}{3T} = \frac{X''}{X} = k$$

$X''=kX \quad X'(0)=0 \quad X(2)=0$

$k<0$ $\quad X = A\cos\sqrt{|k|}\,x + B\sin\sqrt{|k|}\,x$

$X' = -A\sqrt{|k|}\sin\sqrt{|k|}\,x + B\sqrt{|k|}\cos\sqrt{|k|}\,x$

$X'(0) = B\sqrt{|k|} = 0, \qquad B=0$

$X(2) = A\cos\sqrt{|k|}\,2 = 0$

$\sqrt{|k|}\,2 = \left(\frac{1}{2}+n\right)\pi \qquad n=0,1,2,\ldots$

$$K = \frac{-(\frac{1}{2}+n)^2 \pi^2}{4}$$

Now $\quad T' = -\frac{3}{4}(\frac{1}{2}+n)^2 \pi^2 T$

$$T = e^{-\frac{3}{4}(\frac{1}{2}+n)^2 \pi^2 t}$$

General solution is

$$v(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{(\frac{1}{2}+n)\pi x}{2} \, e^{-\frac{3}{4}(\frac{1}{2}+n)^2 \pi^2 t}$$

$$v(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{(\frac{1}{2}+n)\pi x}{2} = \cos \pi x - 1$$

$$A_n = \int_0^2 (\cos \pi x - 1) \cos\frac{(\frac{1}{2}+n)\pi x}{2} \, dx = \frac{64(-1)^n}{\pi(2n+5)(2n+1)(2n-3)}$$

Solution

$$v(x,t) = \sum_{n=0}^{\infty} \frac{64(-1)^n}{\pi(2n+5)(2n+1)(2n-3)} \cos\frac{(\frac{1}{2}+n)\pi x}{2} \, e^{-\frac{3}{4}(\frac{1}{2}+n)^2 \pi^2 t}$$