#1 $u_t = u_{xx} - u + x$
$u(0,t)=0$
$u(1,t)=1$
$u(x,0)=0$

Let $u = w + v$ where w satisfies the boundary $w(0,t)=0$ and $w(1,t)=1$.

$w(x,t) = a(t)x + b(t)$
$w(0,t) = b(t) = 0$ so $b(t) = 0$
$w(1,t) = a(t) = 1$ so $a(t) = 1$
$w(x,t) = x$.

Then v satisfies

$v_t = v_{xx} - v$
$v(0,t)=0$
$v(1,t)=0$
$v(x,0) = -x$

Separation of variables

$XT' = X''T - XT, \qquad XT' + XT = X''T$

$$\frac{T'+T}{T} = \frac{X''}{X} = k$$

Thus $X'' = kX, \quad X(0)=0, \quad X(1)=0$

case $k<0$, $\quad X = A\cos\sqrt{|k|}x + B\sin\sqrt{|k|}x$
$X(0) = A = 0$ so $A = 0$
$X(1) = B\sin\sqrt{|k|} = 0$ so $\sqrt{|k|} = n\pi \quad n = 1, 2, \ldots$

cases $k \geq 0$ has no non-trivial solutions.

$T' + T = -n^2\pi^2 T$
$T' + (1+n^2\pi^2)T = 0$
$T = e^{-(1+n^2\pi^2)t}$

Thus $v(x,t)$ is of the form

$$v(x,t) = \sum_{n=1}^{\infty} B_n e^{-(1+n^2\pi^2)t} \sin n\pi x$$

satisfy the initial condition by choosing B_n

$$\sum_{n=1}^{\infty} B_n \sin n\pi x = -x$$

so

$$B_n = -2\int_0^1 x \sin n\pi x \, dx = 2\,\frac{(-1)^n}{n\pi}$$

It follows that

$$u(x,t) = w(x,t) + v(x,t)$$

$$= x - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} e^{-(1+n^2\pi^2)t} \sin n\pi x$$

#2. $u_{xx} + 4u_{yy} = \sin 2\pi x$

$u(0,y) = 0$

$u(1,y) = 0$

$u(x,0) = 0$

$u(x,1) = \sin \pi x$

Let $u = w + v$ where

$w_{xx} + 4w_{yy} = 0$ | | $v_{xx} + 4v_{yy} = \sin 2\pi x$

$w(0,y) = 0$ | | $v(0,y) = 0$

$w(1,y) = 0$ | and | $v(1,y) = 0$

$w(x,0) = 0$ | | $v(x,0) = 0$

$w(x,1) = \sin \pi x$ | | $v(x,1) = 0$

Solve for w:

$X''Y + 4XY'' = 0$

$\frac{X''}{X} = -\frac{4Y''}{Y} = k$

$X'' = kX \quad , \quad X(0) = 0 \ , \ X(1) = 0$

case $k < 0$

$X = A\cos\sqrt{|k|}\,x + B\sin\sqrt{|k|}\,x$

$X(0) = A = 0$

$X(1) = B\sin\sqrt{|k|} = 0 \qquad \sqrt{|k|} = n\pi, \quad n = 1, 2, 3, \ldots$

case $k \geq 0$ no nontrivial solutions.

$Y'' = \frac{n^2\pi^2}{4} Y \quad , \quad Y(0) = 0$

$Y = Ce^{\frac{n\pi}{2}y} + De^{-\frac{n\pi}{2}y}$

$Y(0) = C + D = 0 \qquad C = -D$

$Y = 2C \sinh \frac{n\pi}{2} y$

So series solution is

$$w(x,y) = \sum_{n=1}^{\infty} F_n \sinh \frac{n\pi}{2} y \, \sin n\pi x$$

To satisfy boundary

$$\sum_{n=1}^{\infty} F_n \sinh\frac{n\pi}{2} \sin n\pi x = \sin \pi x$$

implies $\quad F_1 \sinh\frac{\pi}{2} = 1, \quad F_1 = \dfrac{1}{\sinh(\frac{\pi}{2})}$ and so

$$w(x,y) = \frac{\sinh\frac{\pi}{2}y}{\sinh\frac{\pi}{2}} \sin \pi x$$

Solve for v:

$$v(x,y) = \sum_{n=1}^{\infty} a_n(y) \sin n\pi x$$

so

$$\sum_{n=1}^{\infty} -n^2\pi^2 a_n(y) \sin n\pi x + 4\sum_{n=1}^{\infty} a_n''(y) \sin n\pi x = \sin 2\pi x$$

$$a_n(0) = a_n(1) = 0$$

Gives

$$n \neq 2, \quad -n^2\pi^2 a_n(y) + 4a_n''(y) = 0, \quad a_n(0) = a_n(1) = 0$$

$$n = 2, \quad -4\pi^2 a_2(y) + 4a_2''(y) = 1, \quad a_2(0) = a_2(1) = 0$$

Solving obtains. $a_n = 0$ for $n \neq 2$ and for $n = 2$ and $a_2 = p + h$ where the particular solution

$$p'' - \pi^2 p = 1/4, \qquad p = -\frac{1}{4\pi^2}$$

homogeneous solution

$$h'' - \pi^2 h = 0, \qquad h(0) = \frac{1}{4\pi^2}, \quad h(1) = \frac{1}{4\pi^2}$$

$$h = G\cosh \pi y + H \sinh \pi y$$

$$h(0) = G = \frac{1}{4\pi^2}$$

$$h(1) = \frac{1}{4\pi^2}\cosh \pi + H\sinh \pi = \frac{1}{4\pi^2}$$

$$H \sinh \pi = \frac{1}{4\pi^2}(1 - \cosh \pi)$$

$$H = \frac{1}{4\pi^2 \sinh \pi}(1 - \cosh \pi)$$

Therefore

$$a_1 = p + h = \frac{-1}{4\pi^2} + \frac{1}{4\pi^2}\cosh \pi y + \frac{1}{4\pi^2 \sinh \pi}(1 - \cosh \pi)\sinh \pi y$$

$$a_1 = \frac{1}{4\pi^2}\left(-1 + \cosh \pi y + \frac{\sinh \pi y}{\sinh \pi}(1 - \cosh \pi)\right)$$

and

$$v(x,y) = \frac{-1}{4\pi^2}\left(1 - \cosh \pi y - \frac{\sinh \pi y}{\sinh \pi}(1 - \cosh \pi)\right)\sin 2\pi x$$

Thus

$$u = w + v$$

$$= \frac{\sinh \frac{\pi}{2} y}{\sinh \frac{\pi}{2}}\sin \pi x - \frac{1}{4\pi^2}\left(1 - \cosh \pi y - \frac{\sinh \pi y}{\sinh \pi}(1 - \cosh \pi)\right)\sin 2\pi x$$

#3. $$u'' + q^2 u = f$$
$$u'(0) = u'(3) = 0$$

Find the Green's function.

$$g'' + q^2 g = -\delta(x - x_0)$$
$$g'(0) = g'(3) = 0$$

Thus

$$g(x, x_0) = \begin{cases} C\, u_\ell(x)\, u_r(x_0) & \text{for } x \le x_0 \\ C\, u_\ell(x_0)\, u_r(x) & \text{for } x \ge x_0 \end{cases}$$

where

$$u_\ell'' + q^2 u_\ell = 0, \quad u_\ell'(0) = 0$$

$$u_\ell = A\cos qx + B\sin qx$$
$$u_\ell' = -Aq\sin qx + Bq\cos qx$$
$$u_\ell'(0) = Bq = 0, \quad B = 0$$

For simplicity set $A = 1$

$$u_\ell = \cos qx$$

and

$$u_r'' + q^2 u_r = 0 \qquad u_r'(3) = 0$$
$$u_r = a\cos q(x-3) + b\sin q(x-3)$$
$$u_r' = -aq\sin q(x-3) + bq\cos q(x-3)$$
$$u_r'(3) = bq = 0, \quad b = 0$$

For simplicity set $a = 1$

$$u_r = \cos q(x-3)$$

Therefore

$$g(x, x_0) = \begin{cases} C\cos qx \cos q(x_0 - 3) & x \le x_0 \\ C\cos qx_0 \cos q(x - 3) & x > x_0 \end{cases}$$

The jump condition implies

$$Cq\left(-\sin qx_0 \cos q(x_0-3) + \cos qx_0 \sin q(x_0-3)\right) = 1$$

Since $g(x,x_0) = g(x_0,x)$ then C doesn't depend on x_0. Therefore taking $x_0 = 3$ we obtain

$$Cq(-\sin 3q) = 1$$

$$C = \frac{-1}{q\sin 3q}$$

It follows that

$$g(x,x_0) = \begin{cases} \dfrac{-1}{q\sin 3q} \cos qx \cos q(x_0-3) & \text{for } x \le x_0 \\ \\ \dfrac{-1}{q\sin 3q} \cos qx_0 \cos q(x-3) & \text{for } x \ge x_0 \end{cases}$$

some trigonometry

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Therefore

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b.$$

It follows that

case $x \le x_0$

$$\cos qx \cos q(x_0-3) = \frac{1}{2}\left(\cos q(x+x_0-3) + \cos q(x-x_0+3)\right)$$
$$= \frac{1}{2}\left(\cos q(x+x_0-3) + \cos q(3-|x-x_0|)\right)$$

case $x \ge x_0$

$$\cos qx_0 \cos q(x-3) = \frac{1}{2}\left(\cos q(x_0+x-3) + \cos q(x_0-x+3)\right)$$
$$= \frac{1}{2}\left(\cos q(x+x_0-3) + \cos q(3-|x-x_0|)\right)$$

Since both cases are the same, we can write

$$g(x,x_0) = \frac{-1}{2q\sin 3q}\left(\cos q(x+x_0-3) + \cos q(3-|x-x_0|)\right)$$

#4 $u'' - q^2 u = f$

$u'(0) = u'(3) = 0$

Find Green's function

$$g'' - q^2 g = -\delta(x - x_0)$$

$$g'(0) = g'(3) = 0$$

Thus

$$g(x, x_0) = \begin{cases} C\, u_\ell(x)\, u_r(x_0) & \text{for } x \le x_0 \\ C\, u_\ell(x_0)\, u_r(x) & \text{for } x \ge x_0 \end{cases}$$

where

$u_\ell'' - q^2 u_\ell = 0$, $u_\ell'(0) = 0$

$u_\ell = A \cosh qx + B \sinh qx$

$u_\ell' = Aq \sinh qx + Bq \cosh qx$

$u_\ell'(0) = Bq = 0$ $B = 0$

For simplicity set $A = 1$

$u_\ell = \cosh qx.$

Similarly

$u_r = \cosh q(x-3)$

Therefore

$$g(x, x_0) = \begin{cases} C \cosh qx \cosh q(x_0 - 3) & x \le x_0 \\ C \cosh qx_0 \cosh q(x-3) & x \ge x_0 \end{cases}$$

The jump condition implies

$$Cq\left(\sinh qx_0 \cosh q(x_0-3) - \cosh qx_0 \sinh q(x_0-3)\right) = 1$$

Since C doesn't depend on x_0 ~~then~~ taking $x_0 = 3$

$$Cq \sinh 3q = 1, \qquad C = \frac{1}{q \sinh 3q}$$

It follows

$$g(x,x_0) = \begin{cases} \dfrac{1}{q\sinh 3q} \cosh qx \cosh q(x_0-3) & x \le x_0 \\ \dfrac{1}{q\sinh 3q} \cosh qx_0 \cosh q(x-3) & x \ge x_0 \end{cases}$$

Since

$$\cosh a \cosh b = \frac{e^a+e^{-a}}{2} \cdot \frac{e^b+e^{-b}}{2} = \frac{e^{a+b}+e^{-a-b}+e^{a-b}+e^{-a+b}}{4}$$

$$= \frac{1}{2}\left(\cosh(a+b) + \cos(a-b)\right)$$

Then as in problem #3 we obtain

$$g(x,x_0) = \frac{1}{2q\sinh 3q}\left(\cosh q(x+x_0-3) + \cosh q(3-|x-x_0|)\right)$$

#5a Show $\frac{1}{x^2+1}$ and $\frac{x^5}{x^2+1}$ are solutions to the homogeneous eq

$$(1+x^2)u'' - \frac{4}{x}u' - 6u = 0$$

Let $u = \frac{1}{x^2+1}$, $u' = \frac{-2x}{(x^2+1)^2}$

$$u'' = \frac{-2(x^2+1)^2 + 2x\, 2(x^2+1)\, 2x}{(x^2+1)^4}$$

$$= \frac{-2(x^4+2x^2+1) + 8x^2(x^2+1)}{(x^2+1)^4}$$

$$= \frac{6x^4+4x^2-2}{(x^2+1)^4}$$

Thus

$$(1+x^2)u'' - \frac{4}{x}u' - 6u = \frac{6x^4+4x^2-2}{(x^2+1)^3} - \frac{4}{x}\cdot\frac{-2x}{(x^2+1)^2} - 6\frac{1}{x^2+1}$$

$$= \frac{6x^4+4x^2-2+8(x^2+1)-6(x^4+2x^2+1)}{(x^2+1)^3}$$

$$= \frac{0}{(x^2+1)^3} = 0.$$

Let $u = \frac{x^5}{x^2+1}$, $u' = \frac{5x^4(x^2+1) - x^5\cdot 2x}{(x^2+1)^2}$

$$= \frac{3x^6+5x^4}{(x^2+1)^2}$$

$$u'' = \frac{(18x^5+20x^3)(x^2+1)^{\cancel{2}} - (3x^6+5x^4)2\cancel{(x^2+1)}2x}{(x^2+1)^{\cancel{4}\,3}}$$

$$= \frac{18x^7+38x^5+20x^3 - 12x^7 - 20x^5}{(x^2+1)^3}$$

$$= \frac{6x^7+18x^5+20x^3}{(x^2+1)^3}$$

$$(1+x^2)u'' - \frac{4}{x}u' - 6u = \frac{6x^7+18x^5+20x^3}{(x^2+1)^2} - \frac{4}{x}\,\frac{3x^6+5x^4}{(x^2+1)^2} - 6\,\frac{x^5}{x^2+1}$$

$$= \frac{6x^7+18x^5+20x^3-12x^5-20x^3-6x^7-6x^5}{(x^2+1)^2}$$

$$= \frac{0}{(x^2+1)^2} = 0$$

5b. Write $(x^2+1)u'' - \frac{4}{x}u' - 6u = f$ in the form $(ku')' + pu = F$. What is k, p and F?

Divide by x^2+1 to obtain

$$u'' - \frac{4}{x(x^2+1)}u' - \frac{6}{x^2+1}u = \frac{f}{x^2+1}$$

Integrating factor

$$\mu = e^{-\int \frac{4}{x(x^2+1)}dx}$$

where by partial fractions

$$\frac{-4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$-4 = A(x^2+1) + Bx^2 + Cx$$

$$= (A+B)x^2 + Cx + A$$

Thus $A+B=0$, $C=0$ $A=-4$ and so $B=4$

$$\int \frac{-4}{x(x^2+1)}dx = \int\left(\frac{-4}{x} + \frac{4x}{x^2+1}\right)dx$$

$$= -4\ln x + 2\ln(x^2+1) =$$

$$= 2\left(\ln(x^2+1) - \ln x^2\right)$$

$$= \ln\frac{(x^2+1)^2}{x^4}$$

Therefore

$$\mu = e^{\ln\frac{(x^2+1)^2}{x^4}} = \frac{(x^2+1)^2}{x^4}$$

We obtain that

$$\left(\frac{(x^2+1)^2}{x^4}u'\right)' - \frac{6}{x^2+1}\cdot\frac{(x^2+1)^2}{x^4}u = \frac{f}{x^2+1}\cdot\frac{(x^2+1)^2}{x^4}$$

It follows that

$$K = \frac{(x^2+1)^2}{x^4}$$

$$p = -\frac{6(x^2+1)}{x^4}$$

$$F = \frac{(x^2+1)}{x^4}f$$