#5c Find the Green's function for

$$(1+x^2)u'' - \frac{4}{x}u' - 6u = f$$

$$u(0) = u(1) = 0$$

First find the Green's function for

$$(KG')' + pG = F$$

where K, p and F are as in 5b. Thus

$$G(x,x_0) = \begin{cases} C\, u_\ell(x)\, u_r(x_0) & \text{for } x \le x_0 \\ C\, u_\ell(x_0)\, u_r(x) & \text{for } x \ge x_0 \end{cases}$$

where

$$(1+x^2)u_\ell'' - \frac{4}{x}u_\ell' - 6u_\ell = 0, \quad u_\ell(0) = 0$$

From part 5a we have

$$u_\ell = A\frac{1}{1+x^2} + B\frac{x^5}{1+x^2}$$

$$u_\ell(0) = A = 0$$

Take $B=1$ for convenience.

$$u_\ell(x) = \frac{x^5}{1+x^2}$$

Also $(1+x^2)u_r'' - \frac{4}{x}u_r' - 6u_r = 0, \quad u_r(1) = 0$

$$u_r = a\frac{1}{1+x^2} + b\frac{x^5}{1+x^2}$$

$$u_r'(0) = \frac{a}{2} + \frac{b}{2} = 0 \qquad a = -b$$

Take $a = 1$ for convenience

$$u_r(x) = \frac{1-x^5}{1+x^2}$$

Therefore

$$G(x,x_0)=\begin{cases} C\,\dfrac{x^5}{1+x^2}\cdot\dfrac{1-x_0^5}{1+x_0^2} & \text{for } x\le x_0\\[2ex] C\,\dfrac{x_0^5}{1+x_0^2}\cdot\dfrac{1-x^5}{1+x^2} & \text{for } x\ge x_0\end{cases}$$

The jump condition gives that

$$\cancel{\frac{(x_0^2+1)^2}{\boxed{x_0^4}}}\,C\left(\frac{5\boxed{x_0^4}(1+x_0^2)-x_0^5\,2\cdot x_0}{(1+x_0^2)^2}\cdot\frac{1-x_0^5}{1+x_0^2}\right.$$

$$\left.-\frac{\boxed{x_0^5}}{1+x_0^2}\cdot\frac{-5x_0^4(1-x_0^2)-(1-x_0^5)\,2x_0}{(1+x_0^2)^2}\right)=1$$

Since the problem is self-adjoint $G(x,x_0)=G(x_0,x)$ and so C doesn't depend on x_0. Taking $x_0=0$ we obtain

$$C\left(\frac{5\cdot 1-0}{1}\cdot\frac{1-0}{1}-\frac{0}{1}\cdot\frac{\ast}{1}\right)=1$$

So $C=\frac{1}{5}$. Therefore

$$G(x,x_0)=\begin{cases} \dfrac{1}{5}\,\dfrac{x^5}{1+x^2}\cdot\dfrac{1-x_0^5}{1+x_0^2} & \text{for } x\le x_0\\[2ex] \dfrac{1}{5}\,\dfrac{x_0^5}{1+x_0^2}\cdot\dfrac{1-x^5}{1+x^2} & \text{for } x\ge x_0\end{cases}$$

Therefore

$$u(x) = -\int_0^1 G(x,x_0)\,F(x_0)\,dx_0$$

$$= -\int_0^1 G(x,x_0)\,\frac{x_0^2+1}{x_0^4}\,f(x_0)\,dx_0$$

$$= -\int_0^1 g(x,x_0)\,f(x_0)\,dx_0$$

where

$$g(x,x_0) = G(x,x_0)\,\frac{x_0^2+1}{x_0^4}$$

$$= \begin{cases} \dfrac{1}{5}\,\dfrac{x^5}{1+x^2}\,\dfrac{1-x_0^5}{x_0^4} & \text{for} \quad x \le x_0 \\[2ex] \dfrac{1}{5}\,x_0 \cdot \dfrac{1-x^5}{1+x^2} & \text{for} \quad x \ge x_0\,. \end{cases}$$