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GREEN'S FUNCTIONS AND BOUNDARY VALUE PROBLEMS

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1
Green's Functions (Intuitive Ideas)

1. INTRODUCTION AND GENERAL COMMENTS

For the limited purposes of this section we shall look mainly at steady heat flow in a homogeneous medium. Consider first the one-dimensional problem of a thin rod occupying the interval $(0,1)$ on the x axis. Setting the product of the thermal conductivity and cross-sectional area equal to 1, we find that (1.17), Chapter 0, becomes

$$-\frac{d^2u}{dx^2}=f(x), \quad 0<x<1; \qquad u(0)=\alpha, \quad u(1)=\beta, \tag{1.1}$$

where $f(x)$ is the prescribed source density (per unit *length* of the rod) of heat and α,β are the prescribed end temperatures. The three quantities $\{f(x);\alpha,\beta\}$ are known collectively as the *data* for the problem. The data consists of the *boundary data* α,β and of the *forcing function* $f(x)$.

We shall be concerned not only with solving (1.1) for specific data but also with finding a suitable form for the solution that will exhibit its dependence on the data. Thus as we change the data our expression for the solution should remain useful. The feature of (1.1) that enables us to achieve this goal is its linearity, as reflected in the *superposition principle*: If $u_1(x)$ is a solution for the data $\{f_1(x);\alpha_1,\beta_1\}$ and $u_2(x)$ for the data $\{f_2(x);\alpha_2,\beta_2\}$, then $Au_1(x)+Bu_2(x)$ is a solution for the data $\{Af_1(x)+Bf_2(x);A\alpha_1+B\alpha_2,A\beta_1+B\beta_2\}$. One can extend this principle in an obvious manner to n solutions corresponding to n sets of data. Under mild restrictions it is even possible to extend the superposition principle to infinite sets of data (see Exercise 1.4 for the case of superposition over a continuously varying parameter). In practice, the superposition principle permits us to decompose complicated data into possibly simpler parts, to solve each of the simpler boundary value problems, and then to reassemble these solutions to find the solution of the original problem. One decomposition of the data which is often used is

$$\{f(x);\alpha,\beta\}=\{f(x);0,0\}+\{0;\alpha,\beta\}.$$

The problem with data $\{f(x);0,0\}$ is an *inhomogeneous equation* with *homogeneous* boundary conditions; the problem with data $\{0;\alpha,\beta\}$ is a *homogeneous equation* with *inhomogeneous* boundary conditions. It should be noted that data $\{0;\alpha,\beta\}$ is itself often split up into $\{0;\alpha,0\}$ and $\{0;0,\beta\}$, each of which involves one inhomogeneous and one homogeneous boundary condition.

Later in this section [equation (1.12)] and again in Section 2 [equations (2.9) and (2.10)] we show how the superposition principle or other methods lead to the following form for the solution of (1.1):

$$u(x)=\int_0^1 g(x,\xi)f(\xi)\,d\xi+(1-x)\alpha+x\beta, \tag{1.2}$$

where *Green's function* $g(x,\xi)$ is a function of the real variables x and ξ defined on the square $0\leqslant x,\xi\leqslant 1$ and is explicitly given by

$$g(x,\xi)=x_<(1-x_>)=\begin{cases} x(1-\xi), & 0<x<\xi,\\ \xi(1-x), & \xi<x<1.\end{cases} \tag{1.3}$$

Here $x_<$ stands for the lesser of the two quantities x and ξ, and $x_>$ for the greater of x and ξ. Since g does not depend on the data, it is clear that (1.2) expresses in a very simple manner the dependence of u on the data $\{f;\alpha,\beta\}$. Symbolically we can write (1.2) as

$$u(x)=F(f,\alpha,\beta),$$

where F is a linear operator transforming the data into the solution.

For specific f the integration in (1.2) can sometimes be performed in closed terms, using elementary integration techniques. One must, however, divide the interval of integration into two parts to take advantage of the simple formulas for g. Since the integration in (1.2) is over ξ, we write $\int_0^1=\int_0^x+\int_x^1$, where in the interval from 0 to x we have $\xi<x$, so that we must use the second line of (1.3), whereas in the interval from x to 1 the first line of (1.3) applies. The integral term in (1.2) therefore becomes

$$(1-x)\int_0^x \xi f(\xi)\,d\xi+x\int_x^1(1-\xi)f(\xi)\,d\xi.$$

Turning to the three-dimensional problem of heat conduction in a homogeneous medium of unit thermal conductivity, occupying the domain Ω with boundary Γ, we know, from (1.11), Chapter 0, that the steady temperature $u(x)$ satisfies

$$-\Delta u = f(x), \quad x \in \Omega; \qquad u = h(x), \quad x \in \Gamma. \tag{1.4}$$

Here $x = (x_1, x_2, x_3)$ is a position *vector* in three-dimensional space. The source density per unit volume $f(x)$ is given for $x \in \Omega$, whereas the boundary temperature $h(x)$ is given for x on the surface Γ.

Note that we are no longer using any distinguishing notation for vectors. The context should make it clear whether a quantity is a vector or a scalar. The differential operator Δ appearing in (1.4) is the *Laplacian*, which, in *Cartesian coordinates*, takes the form $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$, whereas in other coordinate systems it will look quite different. One of the advantages of the notation of (1.4) is that it does not commit us to a particular coordinate system.

In any event the solution of (1.4) can be written in terms of Green's function $g(x,\xi)$ (which is now a function of the six real variables $x_1, x_2, x_3, \xi_1, \xi_2, \xi_3$):

$$u(x) = \int_\Omega g(x,\xi) f(\xi)\, d\xi - \int_\Gamma \frac{\partial g}{\partial n} h(\xi)\, dS_\xi, \tag{1.5}$$

where $d\xi$ is an element of volume integration ($= d\xi_1 d\xi_2 d\xi_3$ if Cartesian coordinates are used), dS_ξ is an element of surface integration at the point ξ on Γ, and $\partial/\partial n$ denotes differentiation with respect to ξ in the outward normal direction on Γ.

Thus (1.5) expresses the solution of (1.4) in terms of the data $\{f(x); h(x)\}$ with f the forcing function and h the boundary data; again we see that the superposition principle holds. It remains only to confess that the function $g(x,\xi)$ appearing in (1.5) is usually not known explicitly (unless the domain Ω is of a very simple type such as a ball or parallelepiped); nevertheless one can obtain a great deal of useful information about $g(x,\xi)$. First we point out that $g(x,\xi)$ has a very simple physical interpretation as the temperature at x when the only source is a *concentrated unit source* located at ξ, the boundary being kept at 0 temperature. One can also characterize $g(x,\xi)$ mathematically as the solution of a well-defined boundary value problem; this formulation requires a little delicacy, however, and we shall take up the question in some of the succeeding sections.

The reader may have noticed that in (1.1) the differential equation was formulated on the open interval $0 < x < 1$ rather than on the closed interval

$0 \leqslant x \leqslant 1$; similarly in (1.4) the differential equation held on a *domain* Ω, which, by definition, is an open, connected set (see Section 1, Chapter 0). Why do we insist that Ω be an open set? The reason is to avoid discussing the differential equation on the boundary. Take (1.1), for instance; if we required the differential equation to hold at $x=1$, we either would have to extend the function u for $x>1$ (to be able to form the difference quotient at $x=1$), or would have to use the concept of a one-sided derivative at $x=1$. For a higher dimensional problem such as (1.4), it is even more awkward to try to use the differential equation on the boundary since this would necessarily require some smoothness for the boundary Γ and the boundary data $h(x)$.

We shall therefore always formulate the differential equation on a domain Ω *(open, connected set).*

How do we relate the boundary values of $u(x)$ to its interior values? The boundary values of u are given, whereas the interior values are obtained by solving a differential equation with its attendant indeterminacy. To see that some clarification is needed, consider (1.1) when $f(x) \equiv 0$, $0<x<1$, and $\alpha=\beta=0$. We clearly would like the solution $u(x)$ to be identically 0; we want to rule out ridiculous candidates such as

$$v(x)=\begin{cases} 1, & 0<x<1, \\ 0, & x=0, x=1. \end{cases}$$

This function $v(x)$ satisfies the differential equation $-d^2v/dx^2=0$, $0<x<1$, and clearly $v(0)=v(1)=0$; yet $v(x)$ is a spurious solution. We can reject v by requiring that the solution $u(x)$ of (1.1) be continuous in the *closed interval* $0 \leqslant x \leqslant 1$, or, equivalently, by requiring that $\lim_{x \to 0+} u(x)=\alpha$, $\lim_{x \to 1-} u(x)=\beta$.

Similarly in (1.4) *we shall require that the solution* $u(x)$ *be continuous in the closed region* $\bar{\Omega}=\Omega+\Gamma$. [It is of course understood that the given boundary data $h(x)$ constitutes a continuous function of position on Γ.]

So far we have said nothing about how to decide whether or not a function $u(x)$ satisfies the differential equation $-u''=f(x)$ in (1.1). At first glance there seems to be little to say: one merely makes sure that $u(x)$ is twice differentiable in $0<x<1$ (which implies that u is continuous and has a continuous first derivative) and that the function $-u''(x)$ coincides with the given function $f(x)$ over the whole interval $0<x<1$ [in other words, for each x in $0<x<1$, the numbers $-u''(x)$ and $f(x)$ should be the same]. This works splendidly if $f(x)$ is continuous, but there are good reasons, both mathematical and physical, for considering forcing functions $f(x)$ that are only piecewise continuous. For instance, one can easily envisage a situation in which the prescribed source density $f(x)$ is a nonzero constant, say 1, in $0<x<\xi$ and is 0 in $\xi<x<1$. Note that $f(x)$ is discontinuous at the point $x = \xi$. One often hears the argument that such functions are inadmissible on physical grounds, that the "real" source density is continuous and merely decreases quickly from the value 1 to 0 in a small neighborhood of the point $x = \xi$. Such philosophical arguments are immaterial; all we care about is that the temperature calculated on the basis of the discontinuous source density should be nearly the same (in some suitable sense) as that calculated on the basis of the continuous density (see Exercises 2.2 and 2.6, for instance).

We shall return to this question in due time, but now let us try to incorporate piecewise continuous forcing functions into our framework at the cost of slightly reinterpreting the meaning of the differential equation $-u'' = f$. We still want u' to be an integral of f, and of course integrals of piecewise continuous functions are well defined and are necessarily *continuous*; the continuity of u' implies that u is continuous. The new feature is that u'' no longer exists at the points where f has jumps. Let x_0 be such a point, and let us try to calculate $u''(x_0)$ by forming the difference quotient for u':

$$\frac{u'(x_0+\Delta x) - u'(x_0)}{\Delta x} = \frac{-\int_{x_0}^{x_0+\Delta x} f(x)\,dx}{\Delta x},$$

whose approximate value is $-f(x_0+)$ for $\Delta x > 0$ and $-f(x_0-)$ for $\Delta x < 0$. Thus $u''(x_0)$ cannot exist, no matter how we try to adjust the value of f at x_0, as long as $f(x_0+)$ and $f(x_0-)$ are different. Of course at points where $f(x)$ is continuous we still require that $u''(x)$ exist and satisfy $-u''(x) = f(x)$. We can easily generalize these ideas to an arbitrary linear differential equation of order p:

$$(1.6) \quad a_p(x)u^{(p)}(x) + a_{p-1}(x)u^{(p-1)}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) = f(x), \qquad a < x < b.$$

Definition. Let $f(x)$ be piecewise continuous, and let $a_0(x), \ldots, a_p(x)$ be continuous. A *classical solution* of (1.6) is a function $u(x)$ belonging to $C^{p-1}(a,b)$—the class of functions with continuous derivatives of order $p-1$ on $a < x < b$—such that, at all points of continuity of f, $u^{(p)}(x)$ exists and satisfies the differential equation (1.6).

Remark. By using the notion of weak solution (see Section 5, Chapter 2), we can give a reasonable interpretation of (1.6) even when f is only integrable. This idea applies also to partial differential equations, where difficulties can arise even if f is continuous.

Let us now solve (1.1) for the very simple piecewise continuous, forcing function

$$f(x,a)=H(x-a)=\begin{cases}0, & 0<x<a,\\ 1, & a<x<1,\end{cases} \tag{1.7}$$

where $H(x)$ is the usual Heaviside function, which vanishes for $x<0$ and is equal to 1 for $x>0$ (its value at $x=0$ plays no role in the analysis). In (1.7) x is the primary variable and a is a parameter. We first solve (1.1) when $\alpha=\beta=0$, that is, for data $\{H(x-a);0,0\}$. The solution will be denoted by $u(x,a)$, since it depends not only on x but also on the parameter a. In $0<x<a$ we have $-d^2u/dx^2=0$, whereas in $a<x<1$ we have $-d^2u/dx^2=1$. Integration and use of the boundary conditions gives

$$u=Ax \quad \text{in } (0,a) \qquad \text{and} \qquad u=-\frac{(x-1)^2}{2}+B(1-x) \quad \text{in } (a,1),$$

where A and B may depend on a but not on x. For u to be a classical solution we must require that u and u' be continuous at $x=a$ (we already have more than enough smoothness in the subintervals $0<x<a$ and $a<x<1$). This gives $A=(1-a)^2/2$ and $B=(1-a^2)/2$, so that

$$u(x,a)=\begin{cases}\dfrac{(a-1)^2}{2}x, & 0<x<a,\\[2ex] -\dfrac{(x-1)^2}{2}+\dfrac{1-a^2}{2}(1-x), & a<x<1,\end{cases} \tag{1.8}$$

which is plotted in Figure 1.1. Exercises 1.4 and 1.5 show how to use (1.8) to obtain the solution of (1.1) for *arbitrary* f (with $\alpha=\beta=0$).

It is of interest to present another approach to (1.1) with data $\{f(x);0,0\}$, which lends itself to graphical analysis. This method is based on interpreting the problem as the transverse deflection of a taut string with fixed ends. The static version of (4.36), Chapter 0, with $T^{(0)}=1$, $l=1$, $X=x$, $\mathbf{f}\cdot\mathbf{j}=f(x)$, and u instead of v for the transverse deflection, gives us (1.1) with $\alpha=\beta=0$. It then follows that the vertical component of the tension at a point $(x,u(x))$ along the string is just $u'(x)$; thus the reactions at the ends $x=0$ and $x=1$ are $-u'(0)$ and $u(1)$, respectively. By taking moments about the ends of the string, we find that

$$u'(1)+\int_0^1 xf(x)\,dx=0, \qquad -u'(0)+\int_0^1 (1-x)f(x)\,dx=0, \tag{1.9}$$

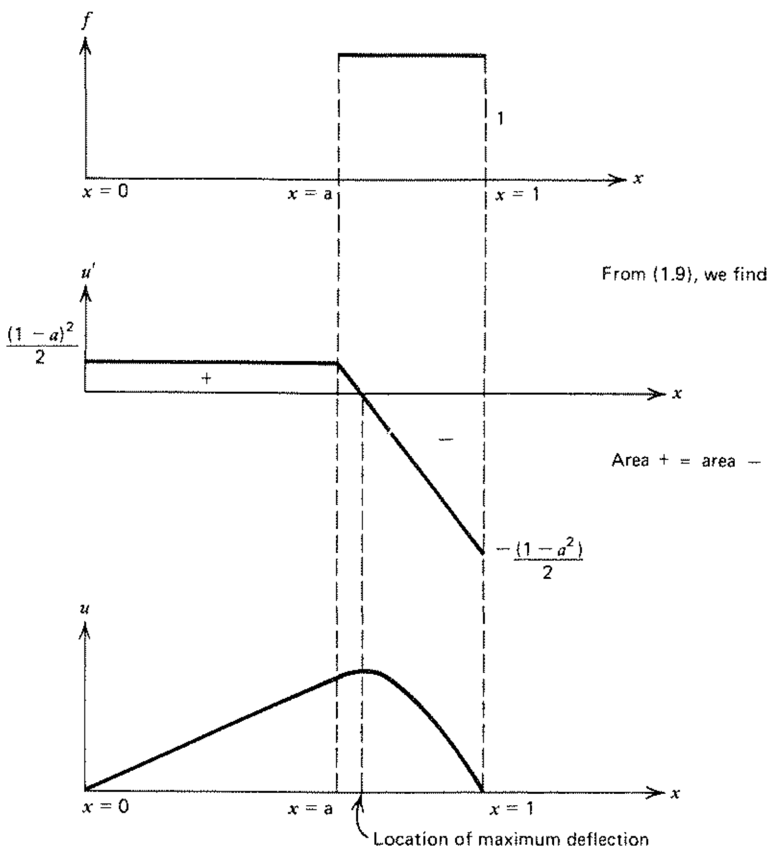

Figure 1.1

which could also be derived without recourse to the physical interpretation by multiplying the differential equation in (1.1) by x and $1-x$, respectively, and then integrating from 0 to 1. In any event we have calculated the reactions at the ends and can now find $u'(x)$ at any point from

$$u'(x)=u'(0)-\int_0^x f(\xi)\,d\xi, \tag{1.10}$$

which can of course be done graphically. Since $u(0)=0$, we can find $u(x)$ from (1.10) by integrating from 0 to x. This again is easy to do graphically; analytically we find that

$$u(x)=\int_0^x u'(\eta)\,d\eta=u'(0)x-\int_0^x d\eta\int_0^\eta f(\xi)\,d\xi. \tag{1.11}$$

The iterated integral can be viewed as a double integral over a triangular region in the ξ-η plane; on changing the order of integration, we obtain

$$\begin{aligned} u(x) &= u'(0)x - \int_0^x (x-\xi)f(\xi)\,d\xi \\ &= x\int_0^1 (1-\xi)f(\xi)\,d\xi - \int_0^x (x-\xi)f(\xi)\,d\xi \qquad (1.12) \\ &= \int_0^1 g(x,\xi)f(\xi)\,d\xi, \end{aligned}$$

where $g(x,\xi)$ is just Green's function as predicted in (1.3). It is then an easy matter to show that (1.2) holds when the data is $\{f;\alpha,\beta\}$ instead of $\{f;0,0\}$. In Figure 1.1 we have illustrated the graphical integration when the data is $\{H(x-a);0,0\}$, the corresponding formula for the deflection being (1.8).

Exercises

1.1. Consider the transverse deflection $u(x)$ of a string satisfying

$$-u''=f(x), \quad 0<x<1; \qquad u(0)=0, \quad u(1)=0,$$

where

$$f(x)=\begin{cases} x-\frac{1}{4}, & 0<x<\frac{1}{2}, \\ \frac{1}{4}, & \frac{1}{2}<x<1. \end{cases}$$

(a) Find u' at one of the ends, and then carry out graphically two successive integrations to obtain the deflection $u(x)$.

(b) Find $u(x)$ using (1.2) and (1.3). To perform the integration explicitly you must divide the interval into $(0,x)$ and $(x,1)$; in the first subinterval, x is larger than ξ, so that the second line of (1.3) applies. You will then need a further subdivision to handle our specific forcing function. Compare your result with that for part (a).

1.2. The small transverse deflection $u(x)$ of a homogeneous beam of unit length subject to a distributed transverse loading $f(x)$ satisfies

$$\frac{d^4u}{dx^4}=f(x), \qquad 0<x<1, \qquad (1.13)$$

where we have set $EI=1$ in (4.11), Chapter 0. For a beam simply supported at its ends the boundary conditions are

$$u(0)=u''(0)=u(1)=u''(1)=0. \qquad (1.14)$$

The shear force V and moment M at a cross section satisfy

$$V=-u'''(x), \qquad M=u''(x),$$

where the choice of signs is in accord with the convention used in Section 4, Chapter 0. For (1.13) subject to (1.14), show how to calculate $u'''(0)$. It is therefore straightforward to find $V(x)$ and $M(x)$ by graphical integration. Once $M(x)$ is known, it is easy to calculate $u'(0)$ and hence to proceed in determining $u'(x)$ and $u(x)$ graphically.

1.3. Consider the boundary value problem

$$(1.15) \quad -\frac{d}{dx}\left[k(x)\frac{du}{dx}\right]=f(x), \quad 0<x<1; \qquad u(0)=u(1)=0.$$

where $k(x)>0$ in $0\leqslant x\leqslant 1$. Let $K(x)$ be a solution of the homogeneous equation satisfying the boundary condition at $x=1$. Show how one can calculate $u'(0)$ by multiplying both sides of the differential equation in (1.15) by $K(x)$ and integrating from $x=0$ to $x=1$.

1.4. For each θ, $\theta_1<\theta<\theta_2$, denote by $u(x,\theta)$ the solution of the problem

$$-\frac{d^2u}{dx^2}=f(x,\theta), \quad 0<x<1; \qquad u|_{x=0}=\alpha(\theta), \quad u|_{x=1}=\beta(\theta).$$

Show that the function

$$U(x)=\int_{\theta_1}^{\theta_2}u(x,\theta)\,d\theta$$

satisfies

$$-\frac{d^2U}{dx^2}=F(x), \quad 0<x<1; \qquad U|_{x=0}=A, \quad U|_{x=1}=B,$$

where

$$F(x)=\int_{\theta_1}^{\theta_2}f(x,\theta)\,d\theta, \qquad A=\int_{\theta_1}^{\theta_2}\alpha(\theta)\,d\theta, \qquad B=\int_{\theta_1}^{\theta_2}\beta(\theta)\,d\theta.$$

1.5. If f has a continuous derivative on $-\infty<x<\infty$, we can write

$$f(x)=f(x_0)+\int_{x_0}^{x}f'(\xi)\,d\xi, \qquad -\infty<x<\infty.$$

For $x > x_0$ show that we can use the equivalent formula

$$(1.16) \qquad f(x) = f(x_0)H(x-x_0) + \int_{x_0}^{\infty} H(x-\xi) f'(\xi)\, d\xi.$$

Use Exercise 1.4 and (1.8) and (1.16) to find the solution of (1.1) with $\alpha = \beta = 0$ in the form (1.2).

2. THE FINITE ROD

Construction of Green's Function

We return to the heat conduction problem (1.1), repeated below for convenience:

$$(2.1) \qquad -\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1; \qquad u(0) = \alpha, \quad u(1) = \beta.$$

We want to solve the problem as compactly as possible for arbitrary data $\{f; \alpha, \beta\}$. The differential operator and the boundary operators appearing on the left sides of the equality signs in (2.1) are kept fixed; no one is proposing to solve all differential equations with arbitrary boundary conditions at one stroke!

To solve (2.1) for arbitrary data, we introduce an accessory problem where, instead of a distributed density of sources, there is only a concentrated source of unit strength at $x = \xi$ and where the *boundary data vanishes* (which means in our case that the temperature is 0 at both ends). Physically this accessory problem makes sense, and the resulting steady temperature should be well defined; moreover, it is clear that the temperature cannot vanish identically, since there is a steady nonzero heat input from the source. This temperature (solution of the accessory problem) is known as Green's function and is denoted by $g(x, \xi)$. Here ξ is the position of the source, and x is the observation point. We usually regard ξ as a parameter and x as the running variable; but when we are all through we have a function of two real variables, and we are at liberty to forget the original significance of x and ξ. In any event all differentiations below are with respect to the first variable in g. Let us see whether we can construct g on the basis of the information available so far. Since there are no sources in $0 < x < \xi$ and in $\xi < x < 1$, we have $-g'' = 0$ in both intervals. Taking into account the fact that g vanishes at $x = 0$ and $x = 1$, we find that

$$(2.2) \qquad g = Ax, \quad 0 < x < \xi; \qquad g = B(1-x), \quad \xi < x < 1.$$

Here A and B are "constants," that is, independent of x; they may, however, depend on the parameter ξ. If at this stage we demanded the continuity of g and g' at $x=\xi$, we would find $A=B=0$, so that g would vanish identically—which is nonsense! We must *abandon* at $x=\xi$ the requirement that g' be continuous, although we shall still insist on the continuity of g. The jump of g' at $x=\xi$ is easily calculated if we recall the primary integral formulation of the problem of heat conduction in terms of energy balance [see (1.1), and (1.10), Chapter 0]. Consider a thin slice of the rod from $\xi-\varepsilon$ to $\xi+\varepsilon$. The one-dimensional character of the problem means that no heat flows through the lateral surface; since the product of the cross-sectional area and the thermal conductivity is 1 and the amount of heat generated in the slice is 1, we have

$$-g'|_{x=\xi+\varepsilon}+g'|_{x=\xi-\varepsilon}=1,$$

which, as ε tends to 0, leads to the *jump condition* for g':

$$g'|_{x=\xi+}-g'|_{x=\xi-}=-1. \tag{2.3}$$

Condition (2.3) and the continuity of g at $x=\xi$ enable us to calculate A and B in (2.2) from the simultaneous equations $A=B(1-\xi)$ and $-B-A=-1$. Thus $B=\xi$ and $A=1-\xi$, so that

$$g(x,\xi)=\begin{cases}(1-\xi)x, & 0\leqslant x<\xi,\\ (1-x)\xi, & \xi<x\leqslant 1,\end{cases} \tag{2.4}$$

confirming (1.3). In Figure 2.1*a* we picture Green's function as a function of x for fixed ξ, and in Figure 2.1*b* as a function of x and ξ. Thus Figure 2.1*a* can be viewed as a cross section of the surface in Figure 2.1*b*.

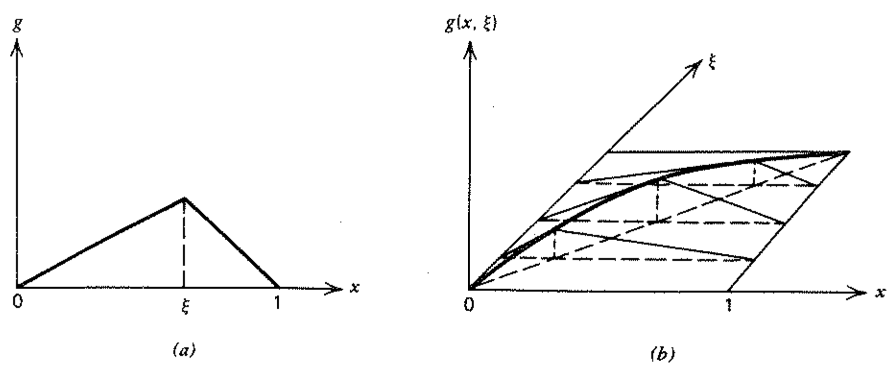

Figure 2.1

We have therefore characterized Green's function $g(x,\xi)$ both physically and mathematically. Before proceeding to another characterization, based on the delta function, let us recapitulate what has been done so far.

1. *Physical description*. We chose to describe g in terms of heat conduction in a rod: $g(x,\xi)$ is the temperature at x when the only source is a unit concentrated source at ξ, the ends being at 0 temperature. It is also possible to interpret g as the transverse deflection of a string: $g(x,\xi)$ is the deflection at x when the only load is a unit concentrated force at ξ, the ends being kept fixed on the x axis at $x=0$ and $x=1$.

2. *Classical mathematical formulation*. Green's function $g(x,\xi)$ associated with (2.1) satisfies

$$(2.5)\qquad \begin{cases} -\dfrac{d^2g}{dx^2}=0, \quad 0<x<\xi,\ \xi<x<1; \\ g(0,\xi)=g(1,\xi)=0; \\ g \text{ continuous at } x=\xi; \qquad \left.\dfrac{dg}{dx}\right|_{x=\xi+}-\left.\dfrac{dg}{dx}\right|_{x=\xi-}=-1. \end{cases}$$

In our third formulation we would like to consider (2.5) as a boundary value problem of the form (2.1) with specific data. The boundary data for g clearly vanishes, but what is the forcing function? In (2.1) the forcing function is a source density (per unit length) rather than the concentrated source of the Green's function problem. How can we describe a concentrated source at the point ξ as a density? This is easy to do symbolically but is not so easy within a consistent mathematical framework. Suppose we let $\delta(x)$ be the density corresponding to a concentrated source at $x=0$. We would then need

$$\int_a^b \delta(x)\,dx=\begin{cases} 1 & \text{if } (a,b) \text{ contains the origin,} \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately, no integrable function satisfies these properties. Nevertheless we shall use $\delta(x)$ symbolically to represent the source density corresponding to a unit source at the origin. This symbolic function is known as the *Dirac delta function*; $\delta(x-\xi)$ is $\delta(x)$ translated ξ units to the right and so must be the source density for a unit source at $x=\xi$.

Perhaps the most natural way to view $\delta(x)$ is as the limit of a sequence of narrow, uniform densities of large magnitude (with total strength unity) such as

$$f_n(x)=\begin{cases} n, & |x|<\frac{1}{2n}, \\ 0, & |x|>\frac{1}{2n}. \end{cases}$$

Thus we may think of $\delta(x)$ as the limit as $n\to\infty$ of $f_n(x)$, and $\delta(x-\xi)$ as the limit of $f_n(x-\xi)$. The *sifting property*

$$\int_a^b \delta(x-\xi)\phi(x)\,dx=\begin{cases} \phi(\xi) & \text{if } a<\xi<b, \\ 0 & \text{if } \xi<a \text{ or } \xi>b, \end{cases} \tag{2.6}$$

where $\phi(x)$ is an arbitrary function continuous at $x=\xi$, then follows by replacing $\delta(x-\xi)$ by $f_n(x-\xi)$ and proceeding to the limit as $n\to\infty$.

3. *Delta function formulation.* Green's function $g(x,\xi)$ associated with (2.1) satisfies

$$-\frac{d^2g}{dx^2}=\delta(x-\xi),\quad 0<x<1,\quad 0<\xi<1;\quad g(0,\xi)=g(1,\xi)=0. \tag{2.7}$$

At this stage (2.7) is nothing but shorthand for (2.5), but we will develop in Chapter 2 a mathematical framework in which (2.7) will have impeccable standing in its own right.

Solution of the Inhomogeneous Equation

The simple physical interpretation for Green's function guides us in constructing the solution of problem (2.1) with data $\{f;0,0\}$:

$$-u''=f(x),\quad 0<x<1;\qquad u(0)=u(1)=0. \tag{2.8}$$

The idea is to decompose the distributed source $f(x)$ into a number of small concentrated sources located at various points along the rod and then add their individual contributions to the temperature to find u. Divide the interval $(0,1)$ into n equal parts, calling the center of the kth subinterval ξ_k. The length of each subinterval is $\Delta\xi=1/n$. It is reasonable to suppose that the temperature corresponding to the distributed density $f(x)$ is closely approximated by the temperature corresponding to small concentrated sources $f(\xi_1)\Delta\xi,\ldots,f(\xi_n)\Delta\xi$, located at $\xi_1,\ldots,\xi_n$, respectively (see Figure 2.2); that is, the temperature for the data $\{f(x);0,0\}$ is close to the temperature for the data $\{\sum_{i=1}^{n}\delta(x-\xi_i)f(\xi_i)\Delta\xi;0,0\}$. According to the principle of superposition extended to concentrated sources, the temperature at x for all the small concentrated sources is

$$\sum_{i=1}^{n} g(x,\xi_i) f(\xi_i) \Delta\xi,$$

which, as $n \to \infty$, tends to

$$u(x) = \int_0^1 g(x,\xi) f(\xi)\, d\xi. \tag{2.9}$$

Thus our intuitive (or *heuristic*) argument leads us to believe that (2.9) provides a solution to (2.8). Observe that this construction will not work directly for (2.1) with nonzero boundary data, but the solution is easy to determine. Since $\alpha(1-x)+\beta x$ satisfies (2.1) with data $\{0;\alpha,\beta\}$, the superposition principle shows that

$$u(x) = \int_0^1 g(x,\xi) f(\xi)\, d\xi + \alpha(1-x) + \beta x \tag{2.10}$$

satisfies (2.1) with data $\{f;\alpha,\beta\}$.

We must now verify that (2.10) *actually solves* (2.1); *we would also like to show that it is the only solution to the problem and, finally, that $u(x)$ depends continuously on the data.*

The rigorous proof below is based, as it must be at this time, on the classical definition (2.5) of g. There will be occasions, however, when we will be satisfied to give merely plausible arguments using the symbolic formulation (2.7), together with the sifting property (2.6) of the delta function.

Verification of Solution

We confine ourselves here to the case where f is continuous, leaving the more general case to Exercise 2.7. Consider first the problem with vanishing boundary data. Clearly (2.9) vanishes at $x=0$ and 1 because $g(0,\xi)=g(1,\xi)=0$, so that we only have to show that $-u''=f$ at each point x,

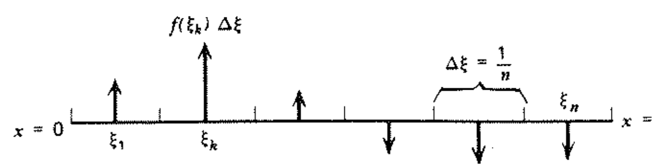

Figure 2.2

$0<x<1$. In view of the discontinuity of g' $(=dg/dx)$ at $x=\xi$, a certain amount of care is required in differentiating expression (2.9). Let us split the interval of integration into the parts $(0,x)$ and $(x,1)$, within each of which g and its derivatives are continuous. Then

$$\frac{du}{dx}=\frac{d}{dx}\left[\int_0^x g(x,\xi)f(\xi)\,d\xi+\int_x^1 g(x,\xi)f(\xi)\,d\xi\right],$$

and we now appeal to the classical formula for differentiation under the integral sign

$$(2.11)\quad \frac{d}{dx}\int_{a(x)}^{b(x)} h(x,\xi)\,d\xi=\int_{a(x)}^{b(x)}\frac{\partial h}{\partial x}\,d\xi+h(x,b(x))\frac{db}{dx}-h(x,a(x))\frac{da}{dx}$$

to obtain

$$\frac{du}{dx}=\int_0^x g'(x,\xi)f(\xi)\,d\xi+\int_x^1 g'(x,\xi)f(\xi)\,d\xi$$
$$+g(x,x-)f(x-)-g(x,x+)f(x+).$$

Here the notation $x-,x+$ serves to distinguish between left- and right-hand values at a possible point of discontinuity of a function. Since $g(x,\xi)$ and $f(\xi)$ are continuous, the distinction is unnecessary in the expression for du/dx, in which the last two terms cancel. A further differentiation leads to

$$\frac{d^2u}{dx^2}=\int_0^x g''f(\xi)\,d\xi+\int_x^1 g''f(\xi)\,d\xi+g'(x,x-)f(x-)-g'(x,x+)f(x+).$$

The jump property (2.3) of g' can be rewritten as $g'(x,x-)-g'(x,x+)=-1$, and since $g''=0$ in the intervals $\xi<x$ and $\xi>x$, it follows that $-u''=f$ and so (2.9) is a solution of (2.8). Since $\alpha(1-x)+\beta x$ satisfies (2.1) with data $\{0;\alpha,\beta\}$, we conclude that (2.10) is a solution of (2.1) as required.

Uniqueness

Suppose $u_1(x)$ and $u_2(x)$ satisfy (2.1) for the same data $\{f(x);\alpha,\beta\}$. Then $w(x)=u_1(x)-u_2(x)$ satisfies (2.1) with data $\{0;0,0\}$. By definition of the concept of a classical solution, u_1' and u_2' must be continuous and u_1'' and u_2'' exist except at points of discontinuity of f. It follows that w and w' are continuous on $0\leqslant x\leqslant 1$ and that $w''=0$ except possibly at points of discontinuity of f. In each subinterval where f is continuous, w' must be constant; since w' is continuous, the constant is the same in each subinterval and therefore $w=Cx+D$ in $0<x<1$. Applying the boundary conditions $w(0)=w(1)=0$, we find that $w\equiv 0$ in $0\leqslant x\leqslant 1$.

It is perhaps worth noting that a similar argument shows also that Green's function satisfying (2.5) is unique.

Continuity with Respect to the Data

In most experimental situations the data $\{f(x);\alpha,\beta\}$ is not known precisely. It would be comforting to know that the solution of the boundary value problem is not hypersensitive to small changes in the data. We feel that many physical problems should exhibit this kind of stability. We would like to show that a "small" change in the data leads only to a "small" change in the solution. To make this precise we must introduce a notion of "separation" or "distance" between functions (for real numbers there is no problem: the distance between a and b is $|b-a|$). Here we shall define two different *numerical* measures of the "distance" between functions (this notion was introduced in Section 6, Chapter 0, and will be treated in greater generality in Chapter 4):

$$d_\infty(f_1,f_2)=\sup_{0\leqslant x\leqslant 1}|f_1(x)-f_2(x)| \tag{2.12}$$

and

$$d_1(f_1,f_2)=\int_0^1|f_1(x)-f_2(x)|\,dx. \tag{2.13}$$

Thus d_∞ is the largest deviation in ordinates between f_1 and f_2, whereas d_1 is the area between the curves f_1 and f_2. Although f_1 and f_2 are functions, d_1 and d_∞ are nonnegative real numbers. In Figure 6.2, Chapter 0, both $d_1(f_1,f_2)$ and $d_\infty(f_1,f_2)$ are small, whereas in Figure 6.3 $d_1(f_1,f_2)$ is small, but not $d_\infty(f_1,f_2)$. Now let f_1 and f_2 be two continuous functions satisfying $d_\infty(f_1,f_2)<\varepsilon$, and let u_1 and u_2 be the corresponding solutions of (2.8). Then

$$|u_1(x)-u_2(x)|=\left|\int_0^1 g(x,\xi)[f_1(\xi)-f_2(\xi)]\,d\xi\right|\leqslant\int_0^1|g(x,\xi)||f_1-f_2|\,d\xi$$
$$\leqslant d_\infty(f_1,f_2)\int_0^1|g(x,\xi)|\,d\xi.$$

Since $\sup_{0\leqslant x,\xi\leqslant 1}|g|=\frac{1}{4}$, it follows that

$$d_\infty(u_1,u_2)=\sup_{0\leqslant x\leqslant 1}|u_1(x)-u_2(x)|\leqslant\tfrac{1}{4}d_\infty(f_1,f_2)\leqslant\frac{\varepsilon}{4},$$

so that the solution of (2.8) depends continuously on the data. Similar calculations show that there is continuous dependence on the data if d_1 is used as a measure of the distance between functions. For (2.1) with nonzero α and β the situation is similar: if $d_\infty(f_1,f_2)<\varepsilon$, $|\alpha_1-\alpha_2|<\varepsilon$, and $|\beta_1-\beta_2|<\varepsilon$, then

$$d_\infty(u_1,u_2)\leqslant \tfrac{1}{4}d_\infty(f_1,f_2)+|\alpha_1-\alpha_2|+|\beta_1-\beta_2|\leqslant \tfrac{9}{4}\varepsilon,$$

and again there is continuous dependence on the data.

When dealing later with more general boundary value problems (or other equations such as integral equations), we shall still be faced with these three questions:

1. Is there at least one solution (*existence*)?
2. Is there at most one solution (*uniqueness*)?
3. Does the solution depend continuously on the data?

If the answer to this trio of questions is affirmative, the problem is said to be *well posed* (otherwise, *ill posed*). Until recently it was sound dogma to require that every "real" physical problem be well posed. However, it is now understood that ill-posed problems occur frequently in practice but that their physical interpretation and mathematical solution are somewhat more delicate.

Alternative Derivations for the Problem with Nonzero Boundary Data

There is no difficulty in visualizing the role of Green's function in solving the problem with data $\{f(x);0,0\}$. We proceed from (2.8) to (2.9) by straightforward, albeit intuitive, arguments. The extra terms in (2.10) corresponding to nonzero boundary data were obtained by a different procedure. Could we have used Green's function for this purpose as well? One way of doing this is by translating the problem with data $\{0;\alpha,\beta\}$ into a problem with nonzero f and vanishing boundary data. Consider the boundary value problem

$$-u''=0,\quad 0<x<1;\qquad u(0)=\alpha,\quad u(1)=\beta, \tag{2.14}$$

and let $h(x)$ be *any function* (not necessarily satisfying any related differential equation) such that $h(0)=\alpha, h(1)=\beta$. Setting

$$u=h+v,$$

we see that v satisfies

$$-v''=h'',\quad 0<x<1;\qquad v(0)=v(1)=0,$$

whose solution by (2.9) is

$$v(x)=\int_0^1 g(x,\xi)h''(\xi)\,d\xi=\int_0^1 g(\xi,x)h''(\xi)\,d\xi,$$

where we have used the symmetry of g, that is, $g(x,\xi)=g(\xi,x)$. Splitting the interval of integration into $(0,x)$ and $(x,1)$, we obtain, after two integrations by parts,

$$v(x)=-g'(\xi,x)h(\xi)\Big|_{\xi=0}^{\xi=x-}-g'(\xi,x)h(\xi)\Big|_{\xi=x+}^{\xi=1},$$

or, using the jump condition on dg/dx given in (2.5),

$$v(x)=-h(x)-g'(1,x)\beta+g'(0,x)\alpha.$$

Since $u=h+v$, we find that

$$u(x)=\alpha g'(0,x)-\beta g'(1,x)=\alpha(1-x)+\beta x, \tag{2.15}$$

in accord with (2.10). Observe that $h(x)$ has disappeared from the final expression (2.15) for u.

Another way of arriving at (2.15) is to combine the differential equation of (2.14) with that for Green's function in the subintervals $(0,\xi)$ and $(\xi,1)$. Since $g''=0$ in each subinterval, we have $ug''-gu''=0$ in $(0,\xi)$ and in $(\xi,1)$, so that

$$\int_0^\xi (ug''-gu'')\,dx+\int_\xi^1 (ug''-gu'')\,dx=0.$$

The relation

$$ug''-gu''=(ug'-gu')', \tag{2.16}$$

which is valid classically in each of the subintervals $(0,\xi)$ and $(\xi,1)$, and the jump condition on g' then yield

$$u(\xi)=u(0)g'(0,\xi)-u(1)g'(1,\xi), \tag{2.17}$$

which is the same as (2.15).

Both methods used so far are rigorously based on (2.5). An alternative to the second of these methods is based formally on the symbolic characterization (2.7) and on the sifting property (2.6). Multiply (2.14) by g and (2.7) by u, subtract, and integrate from 0 to 1 to obtain

$$u(\xi) = -\int_0^1 (ug'' - gu'')\,dx.$$

We now use (2.16) over the whole interval from 0 to 1; we are entitled to do this because we have accounted for the jump in g' by including the term $\delta(x-\xi)$ in (2.7). Thus $u(\xi) = u(0)g'(0,\xi) - u(1)g'(1,\xi)$ as in (2.17).

There is a lesson worth remembering here. In the classical approach we use only the subintervals in which all functions are well behaved, the term $u(\xi)$ in (2.17) arising from the jump in g' at $x=\xi$. In the symbolic approach we deal with the whole interval at once, the term $u(\xi)$ in (2.17) now arising from the fact that there is a delta function on the right side of the differential equation. *Do not mix the two approaches*!

Eigenfunction Expansion

An apparently different approach to (2.1) is by way of the associated *eigenproblem*

(2.18) $$-u'' = \lambda u, \quad 0 < x < 1; \qquad u(0) = u(1) = 0.$$

Here λ is a complex number regarded as a parameter. Since we are dealing with a homogeneous equation of order 2 with two homogeneous boundary conditions, we might expect that (2.18) has only the trivial solution $u \equiv 0$, $0 \leqslant x \leqslant 1$. It turns out that this is true for most values of λ, but that there are exceptional values of λ, known as *eigenvalues*, for which the boundary value problem (2.18) has nontrivial solutions. These nontrivial solutions are called *eigenfunctions*. Observe that an eigenfunction corresponds to a definite eigenvalue but that to an eigenvalue may be associated more than one independent eigenfunction (it is clear of course that any constant multiple of an eigenfunction is again an eigenfunction corresponding to the same eigenvalue; if u_1 and u_2 are eigenfunctions corresponding to the same λ, then $Au_1 + Bu_2$ is also an eigenfunction corresponding to that λ).

For any complex λ we can easily solve the differential equation in (2.18); imposition of the boundary conditions then shows that nontrivial solutions are possible only for $\lambda_1 = \pi^2, \lambda_2 = 4\pi^2, \ldots, \lambda_n = n^2\pi^2, \ldots$. To the eigenvalue $\lambda_n = n^2\pi^2$ corresponds essentially one eigenfunction $u_n(x) = \sin n\pi x$ (what this means is that every eigenfunction corresponding to λ_n is necessarily of the form Au_n). We observe that eigenfunctions corresponding to different eigenvalues are orthogonal, that is,

(2.19) $$\int_0^1 \sin m\pi x \sin n\pi x\,dx = 0, \qquad m \neq n.$$

If we now multiply the differential equation in (2.1) by $u_n(x)$ and integrate from 0 to 1, we find that

$$-\int_0^1 u'' u_n\,dx = \int_0^1 f u_n\,dx,$$

or, after two integrations by parts and use of the boundary conditions,

$$\lambda_n \int_0^1 u u_n\,dx + n\pi(\beta\cos n\pi - \alpha) = \int_0^1 f u_n\,dx$$

or

$$\int_0^1 u u_n\,dx = \lambda_n^{-1}\left[\int_0^1 f u_n\,dx + n\pi(\alpha - \beta\cos n\pi)\right].$$

The number $\int_0^1 u u_n\,dx$ is just one-half the nth Fourier sine coefficient of $u(x)$, so that we can recover $u(x)$ through

$$u(x) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left[\int_0^1 f u_n\,dx + n\pi(\alpha - \beta\cos n\pi)\right]\sin n\pi x. \tag{2.20}$$

In particular, for problem (2.8) having vanishing boundary data, we find that

$$u(x) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left(\int_0^1 f u_n\,dx\right)\sin n\pi x, \tag{2.21}$$

which can be considered as an alternative representation to (2.9). Comparing the two forms, we deduce the *bilinear representation* of Green's function:

$$g(x,\xi) = \sum_{n=1}^{\infty} \frac{2\sin n\pi x \sin n\pi\xi}{n^2\pi^2}, \tag{2.22}$$

which we will study further in Chapters 6 and 7.

We may regard (2.18) as a problem of type (2.8) with forcing function $\lambda u(x)$; the "solution" is then given by (2.9), which becomes

$$u(x) = \lambda\int_0^1 g(x,\xi)u(\xi)\,d\xi, \qquad 0<x<1. \tag{2.23}$$

Since u appears under the integral sign as well as outside, we have not really solved for $u(x)$. Instead we have shown that (2.18) is equivalent to the *integral equation* (2.23).

Exercises

2.1. Let $f_1(x)$ and $f_2(x)$ be piecewise continuous on $0 \leqslant x \leqslant 1$, and let $u_1(x)$ and $u_2(x)$ be the corresponding solutions of (2.8). Show that the following hold:

(a) If $d_\infty(f_1,f_2)<\varepsilon$, then $d_\infty(u_1,u_2)<\varepsilon/8$, which is a slight improvement of the result in the text.

(b) If $d_1(f_1,f_2)<\varepsilon$, then $d_1(u_1,u_2)<\varepsilon/6$.

(c) If $d_1(f_1,f_2)<\varepsilon$, then $d_\infty(u_1,u_2)<\varepsilon/4$.

The last result shows that, if f_1 is piecewise continuous and f_2 is a reasonable continuous approximation to f_1 (such as in the solid and dashed curves of Figure 4.2), the temperatures corresponding to these source functions are *uniformly* close over the entire interval. Thus the statement made in Section 1 about replacing certain continuous sources by idealized piecewise continuous ones has been substantiated.

2.2. Let $0<a<b<1$, and let

$$q(x)=\begin{cases} q, & a<x<b, \\ 0, & \text{otherwise.}\end{cases}$$

Determine the solution $u(x)$ of (1.1) with data $\{q(x);0,0\}$ by superposition from (1.8). Let $q=1/(b-a)$, and take the limit as $b\to a$. Show that $u(x)$ tends *uniformly* to $g(x,a)$ on $0\leqslant x\leqslant 1$. Thus it is reasonable in this case to approximate a unit concentrated source by a uniformly distributed density (of total strength 1) in a narrow neighborhood of a.

2.3. Let λ be an arbitrary complex number. We shall define the *principal value* of $\sqrt{\lambda}$ as follows. If $\sqrt{\lambda}=0$, then $\lambda=0$; if $\lambda\neq 0$, then λ has a unique representation $\lambda=|\lambda|e^{i\theta}$, $0\leqslant\theta<2\pi$, and the principal value of $\sqrt{\lambda}$ is defined as $|\lambda|^{1/2}e^{i\theta/2}$, where $|\lambda|^{1/2}$ is the *positive square root* of the positive real number $|\lambda|$. Throughout this exercise $\sqrt{\lambda}$ will stand for the principal value just defined (note that as a function of a complex variable $\sqrt{\lambda}$ has a discontinuity on the positive real axis).

(a) The general solution of $-u''=\lambda u$ is $u(x)=A+Bx$ if $\lambda=0$; $u(x)=A\exp(i\sqrt{\lambda}\,x)+B\exp(-i\sqrt{\lambda}\,x)$ (or, alternatively, $u(x)=C\sin\sqrt{\lambda}\,x+D\cos\sqrt{\lambda}\,x$) if $\lambda\neq 0$. Show that only the real values $\lambda_n=n^2\pi^2$ are eigenvalues of (2.18).

(b) Find the eigenvalues and eigenfunctions of

$$-u''=\lambda u,\quad 0<x<1;\qquad u'(0)=u'(1)=0.$$

2.4. Find Green's function $g(x,\xi)$ satisfying

$$\text{(2.24)}\qquad \begin{gathered}\frac{d^4g}{dx^4}=\delta(x-\xi),\quad 0<x,\xi<1;\\ g(0,\xi)=g''(0,\xi)=g(1,\xi)=g''(1,\xi)=0\end{gathered}$$

by a graphical method (see Exercise 1.2). Note that g is the deflection of a simply supported beam with a concentrated unit load at $x=\xi$. What is the equivalent classical formulation of (2.24)?

2.5. A simply supported beam $(0<x<l)$ is subject to the distributed transverse loading

$$f(x)=\begin{cases}0, & \left|x-\dfrac{l}{2}\right|>\varepsilon,\\ p, & \dfrac{l}{2}<x<\dfrac{l}{2}+\varepsilon,\\ -p, & \dfrac{l}{2}-\varepsilon<x<\dfrac{l}{2}.\end{cases}$$

Find the reactions at the ends; plot shear and moment diagrams. Denote the moment in the beam by $M(x,\varepsilon)$, and calculate $\lim_{\varepsilon\to 0}M(x,\varepsilon)$ in the following cases:

(a) p is fixed.
(b) $p=1/\varepsilon$.
(c) $p=1/\varepsilon^2$.

Calculate the limiting deflection corresponding to case (c). What is its physical significance? Formulate the limiting problem as a self-contained mathematical problem without any limiting process.

2.6. Let $\{f_n(x)\}$ be a sequence satisfying the following conditions: $f_n(x)\geqslant 0$ for all x, $f_n(x)=0$ for $|x-t|>1/n$, and $\int_{t-1/n}^{t+1/n}f_n(x)\,dx=1$. Let $u_n(x)$ be the deflection of a string of unit length with fixed ends and unit tension subject to the transverse pressure $f_n(x)$. Draw a graph of $u_n'(x)$ for some large values of n. Show that C_n, the constant slope in $0<x<t-1/n$, approaches $(1-t)$ as $n\to\infty$. Next graph the corresponding $u_n(x)$, and prove that

$$\lim_{n\to\infty}u_n(x)=g(x,t)\qquad\text{uniformly in } 0\leqslant x\leqslant 1,$$

where g is Green's function.

2.7. Suppose $f(x)$ is piecewise continuous on $0 \leqslant x \leqslant 1$. This means that f is continuous except at $a_1,\dots,a_k$, where f has simple jumps of amounts $J_1,\dots,J_k$, respectively. We can write

$$f = [f] + \sum_{i=1}^{k} J_i H(x-a_i), \tag{2.25}$$

where $[f]$ is continuous on $0 \leqslant x \leqslant 1$ and all the jumps in f are accounted for in the sum of Heaviside functions.

We have already proved that, if f is continuous, (2.9) satisfies (2.8). To take care of the piecewise continuous case it is clear [in view of (2.25)] that it is enough to treat the special situation where the loading is $H(x-a)$. Show that in this case (2.9) satisfies (2.8) for all $x \neq a$ and that the deflection has a continuous derivative at $x=a$ (the requirements on a classical solution as defined in Section 1 will then be met).

3. MAXIMUM PRINCIPLE

If $f(x)<0$ in (2.1), we have steady heat conduction with sinks. The temperature $u(x)$ satisfies the *differential inequality*

$$-u''<0, \qquad 0<x<1. \tag{3.1}$$

Since heat is removed at every point of the rod, it is physically clear that the maximum temperature must occur on the boundary (which consists of the two points $x=0$ and $x=1$) and nowhere else. From the geometrical point of view, u, having a positive second derivative, is strictly convex. This again shows that the maximum is on the boundary. The proof is trivial: if u had even a relative maximum at the interior point x_0, then $u'(x_0)=0$ and $u''(x_0) \leqslant 0$, contradicting (3.1).

If instead of the strict inequality (3.1) we know only that

$$-u'' \leqslant 0, \qquad 0<x<1, \tag{3.2}$$

we can still conclude that the maximum of u occurs on the boundary, but now it is possible for the maximum to be also attained in the interior if u is identically constant. We have two versions of the *maximum principle*:

1. *Weak version.* Let u be continuous on $0 \leqslant x \leqslant 1$ and satisfy (3.2). Let the maximum of u on the boundary be M. Then $u(x) \leqslant M$, $0 \leqslant x \leqslant 1$.

2. *Strong version*. Let u be continuous on $0 \leqslant x \leqslant 1$ and satisfy (3.2). Suppose $u(x) \leqslant M$ in $0 \leqslant x \leqslant 1$ and $u(x_0) = M$ at an interior point x_0; then $u(x) \equiv M$ in $0 \leqslant x \leqslant 1$.

The first version makes no prediction as to whether the maximum can occur at interior points as well as on the boundary; in the strong version this is ruled out unless u is identically constant.

Proof of weak version. For $\varepsilon > 0$ set $v(x) = u(x) + \varepsilon x^2$; then v satisfies the strict inequality $-v'' < 0$, $0 < x < 1$, so that the maximum of v is on the boundary and

$$v(x) = u(x) + \varepsilon x^2 \leqslant M + \varepsilon.$$

Thus $u(x) \leqslant M + \varepsilon$ for every $\varepsilon > 0$, and hence $u(x) \leqslant M$.

Proof of strong version. Suppose $u(x_1) < M$, where with no loss of generality we can take $x_1 > x_0$. We will show that this leads to a contradiction by constructing a function v satisfying $-v'' < 0$ in $0 < x < x_1$, $v(0) < M$, $v(x_1) < M$, $v(x_0) = M$. Consider the function

$$z = e^{x - x_0} - 1,$$

which is positive for $x > x_0$, is negative for $x < x_0$, and vanishes at x_0. Now choose ε so that $0 < \varepsilon < [M - u(x_1)]/z(x_1)$, which is clearly possible since $M > u(x_1)$ and $z(x_1) > 0$. Then the function

$$v(x) = u(x) + \varepsilon z(x)$$

satisfies

$$-v'' = -u'' - \varepsilon z'' \leqslant -\varepsilon z'' = -\varepsilon e^{x - x_0} < 0,$$

which is the strict inequality (3.1). But $v(0) < M$, $v(x_1) < M$, and $v(x_0) = M$, contradicting the fact that v must have its maximum on the boundary of the interval $(0, x_1)$.

Remarks

1. If instead of (3.2) we had $-u'' \geqslant 0$, u would be concave and u would satisfy a *minimum* principle. The proof is obtained by noting that $w = -u$ satisfies the maximum principle (in either version).

2. If $u''=0$, then both the maximum and minimum principles apply (the result is trivial in one dimension but not in higher dimensions).

The weak version of these principles is easily extended to higher dimensions. Let Ω be a bounded domain with boundary Γ, and let u be continuous on $\bar{\Omega}$. If $-\Delta u \leqslant 0$ in Ω, then $\max u$ occurs on Γ; if $-\Delta u \geqslant 0$ in Ω, then $\min u$ occurs on Γ; if $\Delta u=0$ in Ω, then $\max u$ and $\min u$ occur on Γ.

The strong version of these principles is extended to higher dimensions in Section 3, Chapter 7.

The weak version by itself leads to the following interesting consequences:

1. The solution of the inhomogeneous problem (1.4) is unique. *Proof*: If u_1 and u_2 are two solutions, then $w=u_1-u_2$ satisfies $-\Delta w=0$ in Ω with $w=0$ on Γ. Since the maximum and the minimum of w on the boundary are both 0, w must be identically 0 in the interior.
2. The data $\{f_1(x); h_1(x)\}$ is said to *dominate* $\{f_2(x); h_2(x)\}$ if $f_1(x) \geqslant f_2(x)$ in Ω and $h_1(x) \geqslant h_2(x)$ on Γ. Suppose $\{f_1; h_1\}$ dominates $\{f_2; h_2\}$; then the corresponding solutions of (1.4) satisfy $u_1(x) \geqslant u_2(x)$. *Proof*: $w=u_2-u_1$ satisfies $-\Delta w \leqslant 0$ in Ω and $w \leqslant 0$ on Γ. By the maximum principle, $\max w$ occurs on Γ; therefore $w(x) \leqslant 0$ in Ω.
3. Exercise 3.2 shows that the solution of (1.4) depends continuously on the data (in the d_∞ sense).

For a comprehensive, yet accessible treatment of maximum principles, the reader should consult the book by Protter and Weinberger.

Exercises

3.1. The equation $-(ku')'+qu=f$ governs steady diffusion in an absorbing medium. Here $u(x)$ is the *concentration* of the diffusing substance measured relative to some ambient value (so that u can be positive or negative), $-k(x)\operatorname{grad} u$ is the diffusion flux vector, $q(x)$ measures the absorption properties, and $f(x)$ is the source density. The effect of the term qu is to try to restore the concentration to its ambient value. The same equation also governs the transverse deflection of a string when there is a springlike resistance (the term qu) to such a deflection. In both cases it is natural to take $k(x)>0$, $q(x) \geqslant 0$, and we shall do so.

(a) Let $v(x)$ be continuous on $a \leqslant x \leqslant b$ and satisfy the strict inequality

$$(3.3) \qquad -(kv')'+qv<0, \qquad a<x<b.$$

Show that v cannot have a *positive* (or even nonnegative) relative maximum at an interior point. The example $v=-\cosh x-1$ on $-1<x<1$ satisfies (3.3) with $k=q=1$ and has a negative maximum at the interior point $x=0$.

(b) Let $u(x)$ be continuous on $a\leqslant x\leqslant b$ and satisfy

$$(3.4)\qquad -(ku')'+qu\leqslant 0,\qquad a\leqslant x\leqslant b.$$

State and prove a weak version of the maximum principle for positive solutions of (3.4).

(c) If the inequality in (3.4) is reversed, a minimum principle is obtained for negative solutions of the inequality $-(ku')'+qu\geqslant 0$. State appropriate principles for solutions of the equation $-(ku')'+qu=0$.

(d) Prove uniqueness and continuous dependence on data for $-(ku')'+qu=f$; $u(0)=\alpha$, $u(1)=\beta$.

3.2. Let the continuous functions f_1 and f_2 satisfy $|f_1(x)-f_2(x)|<\varepsilon$ on a bounded domain Ω, while the continuous functions h_1 and h_2 satisfy $|h_1(x)-h_2(x)|<\varepsilon$ on the boundary Γ of Ω. Show that the corresponding solutions of (1.4) satisfy $|u_1(x)-u_2(x)|<\alpha\varepsilon$, where α is a constant which depends only on Ω.

3.3. Derive a strong maximum principle for solutions of

$$-(ku')'\leqslant 0,\qquad 0<x<1,$$

where $k(x)>0$ in $0\leqslant x\leqslant 1$.

3.4. (a) Derive a strong maximum principle for solutions of

$$-u''+pu'\leqslant 0,\qquad 0<x<1,$$

where $p(x)$ is an arbitrary continuous function. *Hint*: First prove the result for solutions of the strict inequality, and then let $v=u+\varepsilon z$, where $z=\exp[\alpha(x-x_0)]-1$ with α suitably chosen.

(b) Derive a strong minimum principle for solutions of $-u''+pu'\geqslant 0$.

(c) State appropriate principles for solutions of $-u''+pu'=0$.

4. EXAMPLES OF GREEN'S FUNCTIONS

Initial Value Problem

A particle of mass m moves along the u axis under the influence of a force $F(t)$ directed along the axis. The motion of the particle is determined by Newton's law with *initial conditions*:

$$m\frac{d^2u}{dt^2}=F(t),\quad t>0;\qquad u(0)=\alpha,\quad \frac{du}{dt}(0)=\beta. \tag{4.1}$$

If the problem were solved over a finite time interval $(0,T)$, T would play no role in the final result. Therefore we may as well consider the equation on the semi-infinite interval $t>0$.

Green's function $g(t,\tau)$ associated with (4.1) satisfies

$$m\frac{d^2g}{dt^2}=\delta(t-\tau),\quad 0<t,\tau<\infty;\qquad g(0,\tau)=0,\quad g'(0,\tau)=0. \tag{4.2}$$

The function $g(t,\tau)$ is the position of a particle initially at rest at the origin and subject to a unit impulse at time τ. We can regard the impulse as the limiting case of a very large force $X(t)$ acting over a very short period of time from τ to $\tau+\Delta\tau$ such that

$$\int_\tau^{\tau+\Delta\tau}X(t)\,dt=1.$$

Such an impulse will cause an instantaneous unit change in the momentum $m(dg/dt)$ of the particle. Thus (4.2) can be written in the equivalent form

$$\begin{cases} m\dfrac{d^2g}{dt^2}=0,\quad 0<t<\tau,\ t>\tau; & g(0,\tau)=g'(0,\tau)=0,\\ g \text{ continuous at } t=\tau; & m\dfrac{dg}{dt}\Big|_{t=\tau+}-m\dfrac{dg}{dt}\Big|_{t=\tau-}=1. \end{cases} \tag{4.3}$$

Since both initial conditions apply to the interval $(0,\tau)$, we find that $g=0$ until $t=\tau$. The continuity of g and the jump condition on g' give

$$g(t,\tau)=\begin{cases} 0, & 0\leqslant t<\tau,\\ \dfrac{t-\tau}{m}, & t>\tau. \end{cases}$$

The superposition principle can then be applied to the problem with 0 initial data:

$$mu'' = F(t), \quad t>0; \qquad u(0)=0, \quad u'(0)=0$$

with the result

$$u(t) = \int_0^\infty g(t,\tau)F(\tau)\,d\tau = \int_0^t \frac{t-\tau}{m} F(\tau)\,d\tau. \tag{4.4}$$

Not surprisingly, the displacement $u(t)$ is independent of the force acting after time t. The solution of the problem with data $\{0;\alpha,\beta\}$ is $\alpha+\beta t$, so that the solution of (4.1) is the sum of $\alpha+\beta t$ and (4.4). Existence, uniqueness, and continuous dependence on data are easily proved.

Reverting to the x,ξ notation and setting $m=-1$, we see that the function

$$h(x,\xi) = \begin{cases} 0, & 0<x<\xi, \\ \xi - x, & x>\xi, \end{cases} \tag{4.5}$$

satisfies

$$-\frac{d^2h}{dx^2} = \delta(x-\xi), \quad 0<x,\xi; \qquad h(0,\xi)=h'(0,\xi)=0. \tag{4.6}$$

A Green's function for an initial value problem is sometimes called a *causal* Green's function. Green's function $g(x,\xi)$ given by (2.4) satisfies the same differential equation but with different side conditions. With ξ fixed, $h-g$ satisfies $(h-g)''=0$ for all x, and $h-g$ must coincide for all x with a solution of the homogeneous equation, which turns out to be $-(1-\xi)x$. This suggests a method for constructing Green's function for a particular set of boundary conditions: first construct the causal Green's function for the same operator, and then add the appropriate solution of the homogeneous equation to satisfy the original boundary conditions (see the beam problem below, for instance).

Variable Conductivity

Let the thermal conductivity in a rod of unit length be a function $k(x)$ which is positive and continuously differentiable. The steady temperature $g(x,\xi)$ in a rod with a concentrated unit source at ξ, with its left end at 0 temperature, and with its right end insulated satisfies

$$-\frac{d}{dx}\left(k(x)\frac{dg}{dx}\right) = \delta(x-\xi), \quad 0<x,\xi<1; \qquad g(0,\xi)=0, \quad g'(1,\xi)=0. \tag{4.7}$$

An equivalent formulation is

$$(4.8)\quad \begin{cases} -(kg')'=0, \quad 0<x<\xi,\ \xi<x<1; & g(0,\xi)=0, \quad g'(1,\xi)=0, \\ g \text{ continuous at } x=\xi; & k(\xi)[\,g'(\xi+,\xi)-g'(\xi-,\xi)\,]=-1, \end{cases}$$

the jump condition on g' stemming from a heat balance for a thin slice of the rod containing the source. The functions

$$u_1(x)=\int_0^x \frac{1}{k(y)}\,dy \qquad \text{and} \qquad u_2(x)=1$$

are solutions of the homogeneous equation satisfying, respectively, the boundary conditions at the left and right endpoints. The matching conditions at $x=\xi$ give

$$g(x,\xi)=\begin{cases} \displaystyle\int_0^x \frac{1}{k(y)}\,dy, & 0\leqslant x<\xi, \\ \displaystyle\int_0^\xi \frac{1}{k(y)}\,dy, & \xi<x\leqslant 1. \end{cases}$$

Simply Supported Beam

Consider a simply supported beam under a concentrated load at $x=\xi$. The deflection $g(x,\xi)$ satisfies

$$(4.9)\quad \frac{d^4g}{dx^4}=\delta(x-\xi), \quad 0<x,\xi<1; \quad g(0,\xi)=g''(0,\xi)=g(1,\xi)=g''(1,\xi)=0.$$

The shear force $V(x)$ experiences a jump discontinuity from $x=\xi-$ to $\xi+$ to balance the concentrated load:

$$V(\xi+)-V(\xi-)=-1.$$

The moment, the slope, and the deflection remain continuous even at $x=\xi$. Since $-V=d^3g/dx^3$, we can write (4.9) as

$$(4.10)\quad \begin{cases} \dfrac{d^4g}{dx^4}=0, \quad 0<x<\xi,\ \xi<x<1; \\ g(0,\xi)=g''(0,\xi)=g(1,\xi)=g''(1,\xi)=0, \\ g,g',g'' \text{ continuous at } x=\xi; \qquad g'''(\xi+,\xi)-g'''(\xi-,\xi)=1. \end{cases}$$

Applying the boundary conditions, we find that the solution for $x<\xi$ is $Ax+Bx^3$, while for $x>\xi$ it is $C(1-x)+D(1-x)^3$. It remains to apply the matching conditions at ξ. The conditions on g'' and g''' should be used first to yield $g=Ax-(1-\xi)(x^3/6)$ for $x<\xi$ and $g=C(1-x)-\xi[(1-x)^3/6]$ for $x>\xi$. The continuity of g and g' then gives $A=\frac{1}{6}\xi(1-\xi)(2-\xi)$ and $C=\frac{1}{6}\xi(1-\xi)(1+\xi)$. The same result can of course be obtained (perhaps more intuitively) by using the shear and moment diagrams of Exercise 1.2.

We can also construct g by first finding the causal fundamental solution $h(x,\xi)$ satisfying

$$\frac{d^4h}{dx^4}=\delta(x-\xi),\quad 0<x,\xi;\qquad h(0,\xi)=h'(0,\xi)=h''(0,\xi)=h'''(0,\xi)=0.$$

An easy calculation gives

$$h(x,\xi)=\begin{cases}0, & x<\xi,\\ \dfrac{(x-\xi)^3}{6}, & x>\xi.\end{cases}\tag{4.11}$$

Therefore g in (4.10) must be of the form $h+A+Bx+Cx^2+Dx^3$. The conditions at the end $x=0$ give $A=C=0$. At the right end we have $h''(1,\xi)+6D=0$ and $h(1,\xi)+B+D=0$, that is, $D=-(1-\xi)/6$ and $B=\xi(1-\xi)(2-\xi)/6$, which when substituted in $h+Bx+Dx^3$ confirm the earlier result.

The Infinite Rod with Absorption

The steady-state concentration $u(x)$ of a substance diffusing in a homogeneous absorbing medium satisfies

$$-\frac{d^2u}{dx^2}+q^2u=f(x),\qquad -\infty<x<\infty,\tag{4.12}$$

where q^2 is a positive constant, $f(x)$ is the source density of the substance, and the process can be considered as taking place in an infinitely long tube, $-\infty<x<\infty$. (The same equation governs the small transverse displacements of a string subject to an applied load and a springlike restoring mechanism.) Green's function corresponding to a steady unit input of the diffusing substance at $x=\xi$ satisfies

$$-\frac{d^2g}{dx^2}+q^2g=\delta(x-\xi),\qquad -\infty<x,\xi<\infty.\tag{4.13}$$

Since the coefficients of the differential equation are constants, it will suffice to find $g(x,0)$ and then set $g(x,\xi)=g(x-\xi,0)$. This argument obviously depends also on the fact that we are dealing with the infinite domain $-\infty<x<\infty$. Again we assume that $g(x,0)$ is continuous; conservation of matter gives $-g'(0+,0)+g'(0-,0)=1$. In keeping with the absorbing nature of the medium, we require that g *vanish* at $x=\pm\infty$, so that $g(x,0)=e^{-q|x|}/2q$, and

$$g(x,\xi)=\frac{e^{-q|x-\xi|}}{2q}. \tag{4.14}$$

It is perhaps a little surprising that g has no limit as $q\to 0$. The reason is that the nonabsorbing problem cannot obey the condition $g\to 0$ as $|x|\to\infty$. On the other hand, the flux dg/dx obtained from (4.14) has the limits $-\frac{1}{2}$ for $x>\xi$ and $+\frac{1}{2}$ for $x<\xi$, so that, by integration, we might suspect that a solution of (4.13) for $q=0$ is $-(|x-\xi|/2)+C$, which is easily confirmed. Although there is no compelling physical argument for doing so, we often set $C=0$.

Method of Images

Consider (4.13) for the semi-infinite interval $0<x<\infty$. In addition to the condition $g\to 0$ as $x\to\infty$, we now need a boundary condition at $x=0$, which we will take as $g(0,\xi)=0$. This means that any of the diffusing substance that reaches $x=0$ is removed [the boundary condition $g'(0,\xi)=0$ would model a reflecting wall at $x=0$]. Thus we wish to solve

$$\begin{gathered}-\frac{d^2g}{dx^2}+q^2g=\delta(x-\xi), \quad 0<x,\xi<\infty;\\ g(0,\xi)=0, \quad g\to 0 \text{ as } x\to\infty.\end{gathered} \tag{4.15}$$

Let us look instead at an infinite rod with a unit source at $x=\xi$ and a unit sink at $x=-\xi$. According to (4.14), the solution of this problem is

$$\frac{e^{-q|x-\xi|}}{2q}-\frac{e^{-q|x+\xi|}}{2q}. \tag{4.16}$$

This function vanishes at $x=0$ and has only one source singularity in $0<x<\infty$ namely, the original source at $x=\xi$. The term $e^{-q|x+\xi|}/2q$ arising from the *image* source at $x=-\xi$ satisfies the homogeneous differential equation in $0<x<\infty$. Thus (4.16) is a solution of the boundary value problem (4.15).

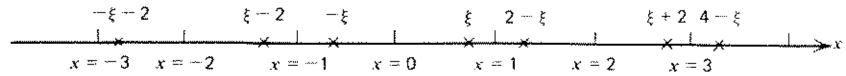

Figure 4.1

For Green's function of a finite rod we use a similar idea. If the ends are both *reflecting*, say, the boundary condition is $dg/dx=0$ at both $x=0$ and $x=1$. We consider the related problem of an infinite rod with *positive* unit sources located at the set of points $\xi+2n$ and $-\xi+2n$, n ranging through the integers from $-\infty$ to ∞, as in Figure 4.1. The solution of this problem is

$$(4.17) \qquad \sum_{n=-\infty}^{\infty} \frac{e^{-q|x-(\xi+2n)|}}{2q} + \sum_{n=-\infty}^{\infty} \frac{e^{-q|x-(-\xi+2n)|}}{2q},$$

which has even symmetry about both $x=0$ and $x=1$. Thus this function has a vanishing derivative at $x=0$ and $x=1$. It is clear from the figure that of this array of sources the only one in the interval $(0,1)$ is the original source. Therefore (4.17) is a solution of the problem of the finite tube with reflecting walls.

Steady Diffusion in a Three-Dimensional Medium

Let Ω be a bounded or unbounded domain in R_3, and let x be the position vector in R_3. The concentration $u(x)$ of the diffusing substance satisfies the partial differential equation

$$(4.18) \qquad -\Delta u + q^2 u = f(x), \qquad x \in \Omega,$$

where the constant $q^2 \geqslant 0$ is a measure of the absorption of the medium and $f(x)$ is the density of the source. There will of course be boundary conditions on Γ, the boundary of Ω. The case $q=0$ corresponds to diffusion without absorption or to steady heat conduction.

Let us look at the case where Ω is the whole space and there is only a concentrated steady unit source at the origin. A mass balance (or heat balance) shows that the flux through a small sphere about the source must equal the input in the ball, that is,

$$(4.19) \qquad \lim_{\varepsilon\to 0} -\int_{|x|=\varepsilon} \frac{\partial u}{\partial n}\, dS = 1.$$

We also expect u to vanish at infinity. The concentration should clearly depend only on the radial coordinate r; since there are no sources for $r\neq 0$, we find, on using the spherical form of Δ, that $u(r)$ satisfies

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)+q^2u=0, \qquad r>0, \tag{4.20}$$

with (4.19) becoming

$$-1=\lim_{\varepsilon\to 0}4\pi\varepsilon^2\left(\frac{du}{dr}\right)_{r=\varepsilon}. \tag{4.21}$$

The substitution $v=u/r$ transforms (4.20) into $-v''+q^2v=0$, whose general solution is a linear combination of e^{-qr} and e^{qr}. Taking account of the required behavior at $r=\infty$, we obtain $u=Ae^{-qr}/r$. Imposing (4.21) gives $A=1/4\pi$, and therefore

$$u=\frac{e^{-qr}}{4\pi r}. \tag{4.22}$$

The effect of a source at ξ is obtained from (4.22) by translation. The concentration due to such a source is what we call the *free space* Green's function:

$$g=\frac{e^{-q|x-\xi|}}{4\pi|x-\xi|}. \tag{4.23}$$

Note that, unlike the one-dimensional case, the limit as $q\to 0$ gives the solution $1/4\pi|x-\xi|$ for a nonabsorbing medium. A more important observation is that Green's function is now *singular* at $x=\xi$ (in one dimension g was continuous at $x=\xi$). This is of course the free space Green's function for the negative Laplacian:

$$-\Delta\frac{1}{4\pi|x-\xi|}=\delta(x-\xi). \tag{4.24}$$

Green's functions for some simple domains (such as a half-space, a quarter-space, a slab, a rectangular parallelepiped) can be found by images when the boundary condition is that the function or its normal derivative vanishes on the boundary. Other methods for constructing Green's functions for partial differential equations will be discussed in later chapters, but suppose for the time being that Green's function $g(x,\xi)$ is known for the negative Laplacian in a domain Ω with $g=0$ on the boundary Γ. Then

g is the solution of

$$-\Delta g = \delta(x-\xi), \quad x, \xi \text{ in } \Omega; \qquad g=0, \quad x \text{ on } \Gamma. \tag{4.25}$$

Let $u(x)$ be the solution of the problem with data $\{f;0\}$; that is, $u(x)$ satisfies

$$-\Delta u = f, \quad x \text{ in } \Omega; \qquad u=0, \quad x \text{ on } \Gamma. \tag{4.26}$$

By the superposition principle we still expect the solution to be expressible as

$$u(x) = \int_\Omega g(x,\xi) f(\xi)\, d\xi. \tag{4.27}$$

Clearly this function vanishes when x is on Γ because g does. If we formally calculate $-\Delta u$ by differentiating under the integral sign in (4.27) and use (4.25), we obtain $-\Delta u = f$ as required. The procedure is permissible if f obeys some very mild restrictions.

Next we express the solution of the problem with data $\{0;h\}$ in terms of Green's function. Let $v(x)$ satisfy

$$-\Delta v = 0, \quad x \text{ in } \Omega; \qquad v = h(x), \quad x \text{ on } \Gamma, \tag{4.28}$$

and multiply the differential equation by g, multiply (4.25) by v, subtract, and integrate over Ω to obtain

Green's theorem

$$v(\xi) = \int_\Omega (g\,\Delta v - v\,\Delta g)\, dx. = \int_\Gamma \left(g\frac{\partial v}{\partial n} - v\frac{\partial g}{\partial n}\right) dS_x$$

By using Green's theorem and the fact that g vanishes for x on Γ, we find that

$$v(\xi) = -\int_\Gamma \frac{\partial g(x,\xi)}{\partial n_x} h(x)\, dS_x$$

or

$$v(x) = -\int_\Gamma \frac{\partial g(\xi,x)}{\partial n_\xi} h(\xi)\, dS_\xi, \tag{4.29}$$

where the subscript indicates the variable of differentiation or integration.

The solution of the problem with data $\{f;h\}$ is then the sum of (4.27) and (4.29), as stated in (1.5), where the symmetry of $g(x,\xi)$ was also used.

Other problems of interest have sources spread on surfaces in R_3. The forcing function here stands somewhere between an ordinary volume density of sources and the most highly concentrated forcing function, $\delta(x-\xi)$, corresponding to a point source. Suppose, for instance, that a layer of sources whose total strength is unity is spread uniformly over the sphere $|x|=a$. The corresponding solution of (4.18) will then depend only on the radial coordinate r measured from the center of the sphere (the differential operator being invariant under rotation). Denoting the solution by $u(r)$, we see, by using the form of Δ appropriate for spherical coordinates, that

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)+q^2u=0, \qquad 0<r<a, \quad r>a. \tag{4.30}$$

We search for a solution which is finite at $r=0$, vanishes as $r\to\infty$, and represents the appropriate source at $r=a$. The total flux on the sphere $|x|=a+\varepsilon$ minus the flux on $|x|=a-\varepsilon$ must equal the input in the interior of the shell, that is,

$$\lim_{\varepsilon\to 0} -\left[\int_{|x|=a+\varepsilon}\frac{\partial u}{\partial n}\,dS+\int_{|x|=a-\varepsilon}\frac{\partial u}{\partial n}\,dS\right]=1.$$

Since u depends only on r, this becomes

$$-1=4\pi a^2\left[\left(\frac{du}{dr}\right)_{r=a+}-\left(\frac{du}{dr}\right)_{r=a-}\right]. \tag{4.31}$$

The solution of (4.30) must therefore satisfy (4.31), vanish at infinity, and be bounded at $r=0$. We find that

$$u=A\frac{\sinh qr}{r} \quad \text{for } r<a, \qquad u=B\frac{e^{-qr}}{r} \quad \text{for } r>a.$$

Since the problem has been reduced to a one-dimensional problem, it is appropriate to require that u be continuous at $r=a$; this condition, together with (4.31), then yields

$$u=\begin{cases}\dfrac{1}{4\pi q}\dfrac{e^{-qa}}{a}\dfrac{\sinh qr}{r}, & r<a,\\[2ex] \dfrac{1}{4\pi q}\dfrac{\sinh qa}{a}\dfrac{e^{-qr}}{r}, & r>a.\end{cases}$$

As $a \to 0$ we should recover the solution $g(x,0)$ for a unit source at the origin. Taking the limit in the expression valid for $r > a$, we find that

$$u = \frac{1}{4\pi} \frac{e^{-qr}}{r},$$

in agreement with (4.22).

In the case of a *line source* of uniform unit density, the response u is independent of the coordinate parallel to the line. It is therefore appropriate to use cylindrical polar coordinates (ρ, ϕ) with the source at $\rho = 0$; the axial symmetry of the problem suggests that u is independent of ϕ and (4.18) reduces to

$$-\frac{1}{\rho} \frac{d}{d\rho}\left(\rho \frac{du}{d\rho}\right) + q^2 u = 0, \qquad \rho > 0.$$

This is the modified Bessel equation whose independent solutions are $I_0(q\rho)$ and $K_0(q\rho)$. Since I_0 is exponentially large at ∞, we must have $u = AK_0(q\rho)$ with A determined from the unit source condition at $\rho = 0$. This condition has the form

$$-1 = \lim_{\varepsilon \to 0} 2\pi\varepsilon \left(\frac{du}{d\rho}\right)_{\rho = \varepsilon},$$

which in light of the logarithmic singularity of K_0 at the origin gives $A = \frac{1}{2}\pi$ and

$$u = \frac{1}{2\pi} K_0(q\rho). \tag{4.32}$$

Note that in three dimensions a concentrated source gives rise to a singularity of order $1/|x|$, a line source to a logarithmic singularity, and a surface source to a simple discontinuity in the normal derivative. Only in the last case is the response continuous across the source.

Interface Problems

Consider steady one-dimensional heat conduction in a rod occupying the interval $-1 < x < 1$. The rod's thermal conductivity is the positive constant k_1 in $-1 < x < 0$ and another positive constant k_2 in $0 < x < 1$. Such a problem could arise in dealing with a *composite rod* constructed by joining end to end two rods of unit length and of different conductivities or in attempting to idealize a *heterogeneous rod* whose conductivity changes rapidly but continuously from k_1 to k_2. In both interpretations we want to reduce the problem to solving constant conductivity equations in the two halves of the rod, and the question that remains is how to match these solutions at the interface $x=0$.

For the heterogeneous rod we are considering the limiting case as $\varepsilon \to 0+$ of a problem with a continuously varying positive conductivity $k(x,\varepsilon)$ having the property

(4.33) $$\lim_{\varepsilon \to 0+} k(x,\varepsilon) = \begin{cases} k_1, & x<0, \\ k_2, & x>0. \end{cases}$$

The limiting conductivity will be denoted by $k(x)$; we have $k(x)=k_1+(k_2-k_1)H(x)$, where H is the Heaviside function. It turns out that it is not quite sufficient to ask that (4.33) hold pointwise. Instead we will need to require that $1/k(x,\varepsilon)$ tend to its limit in the L_1 sense (see Section 7, Chapter 0), that is,

(4.34) $$\lim_{\varepsilon \to 0} \int_{-1}^{1} \left| \frac{1}{k(x,\varepsilon)} - \frac{1}{k(x)} \right| dx = 0,$$

which means that the area between the curves $1/k(x,\varepsilon)$ and $1/k(x)$ must go to 0 as $\varepsilon \to 0$ (this does not follow from pointwise convergence alone). There are many ways of generating explicit expressions for $k(x,\varepsilon)$, for instance,

$$k(x,\varepsilon) = \frac{k_1+k_2}{2} + \frac{k_2-k_1}{\pi} \arctan \frac{x}{\varepsilon},$$

but our results are independent of the particular form of $k(x,\varepsilon)$. The resulting interface conditions are

(4.35) $$k_2 u'(0+) - k_1 u'(0-) = 0$$

and

(4.36) $$u(0+) - u(0-) = 0.$$

The first of these conditions is a consequence of the integral formulation of the law of heat conduction, which states in our case that the heat fluxes to the left and right of $x=0$ must be equal in the absence of concentrated sources at the interface (see Section 1, Chapter 0). The second condition is nearly obvious but does in fact require (4.34), as we shall see in the special case analyzed below.

For a *composite* rod made by joining two rods together, conditions (4.35) and (4.36) are appropriate only if the unit rods are joined perfectly at $x=0$ with no film or gap between them.

Let us now consider the explicitly solvable boundary value problem

$$(4.37)\qquad -\frac{d}{dx}\left(k(x,\varepsilon)\frac{du}{dx}\right)=1, \quad -1<x<1; \qquad u'(-1)=0=u(1).$$

Here we have a heterogeneous rod subject to a uniform density of sources with its left end insulated and its right end kept at 0 temperature. We are interested in the limiting case of ε tending to 0 for k satisfying (4.33) and (4.34); we take $k_2>k_1$ for the sake of definiteness. In view of (4.37) we have $-k(x,\varepsilon)u'=x+1$, so that, by (4.33), the pointwise limit of u' exists, and

$$(4.38)\qquad \lim_{\varepsilon\to 0} u'(x,\varepsilon)=\begin{cases} -\dfrac{x+1}{k_1}, & x<0, \\ -\dfrac{x+1}{k_2}, & x>0. \end{cases}$$

We shall denote the function on the right of (4.38) by $u'(x)$. We observe that $u'(x)$ is discontinuous at $x=0$ and satisfies (4.35) despite the fact that, for each $\varepsilon>0$, $u'(x,\varepsilon)$ is continuous at $x=0$. The situation is illustrated in Figure 4.2. For a fixed small value of ε there is a very sharp change in $u'(x,\varepsilon)$ in a thin transition layer around the interface; the continuity of $u'(x,\varepsilon)$ at $x=0$ is deceptive—the useful information is really contained in the discontinuous function $u'(x)$.

Since $u(x,\varepsilon)=-\int_x^1 u'(\eta,\varepsilon)\,d\eta$, we have

$$(4.39)\qquad u(x,\varepsilon)=\int_x^1 \frac{\eta+1}{k(\eta,\varepsilon)}\,d\eta,$$

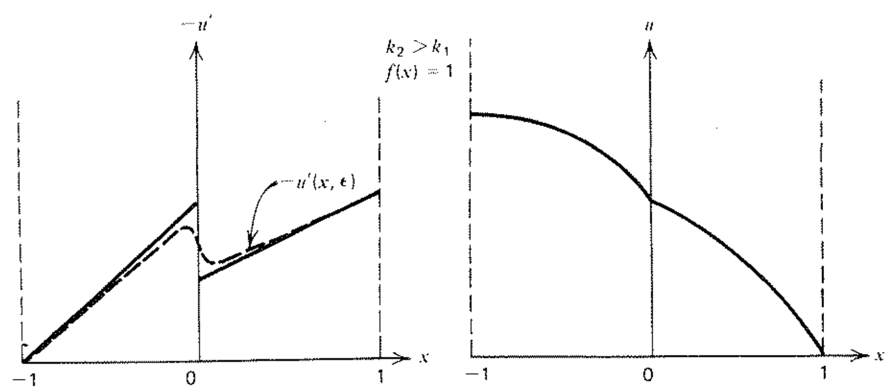

Figure 4.2

where, in view of (4.34), the limit as $\varepsilon \to 0$ may be taken under the integral sign, so that

$$(4.40)\quad u(x) = \begin{cases} \displaystyle\int_x^1 \frac{\eta+1}{k_2}\,d\eta = \frac{3}{2k_2} - \frac{1}{k_2}\left(x + \frac{x^2}{2}\right), & x>0, \\ \displaystyle\int_x^0 \frac{\eta+1}{k_1}\,d\eta + \int_0^1 \frac{\eta+1}{k_2}\,d\eta = \frac{3}{2k_2} - \frac{1}{k_1}\left(x+\frac{x^2}{2}\right), & x<0, \end{cases}$$

which is certainly continuous at $x=0$. The continuity of $u(x)$ is guaranteed from the fact that we can pass to the limit under the integral sign in (4.39), making $u(x)$ the integral of a piecewise continuous function, and such an integral is of course continuous.

Since the interface conditions (4.35) and (4.36) are deduced by a limiting process from a continuously varying conductivity, we shall call them the *natural interface conditions*.

Let us now calculate Green's function $g(x,\xi)$ for a composite rod with natural interface conditions at $x=0$ and the same boundary conditions as in (4.37). We first place the unit source at ξ in the left half of the rod, so that g satisfies

$$(4.41)\quad \begin{cases} -g''=0, \quad -1<x<\xi,\ \xi<x<0,\ 0<x<1; \\ g'(-1,\xi)=0, \quad g(1,\xi)=0; \\ g(\xi+,\xi)=g(\xi-,\xi), \qquad g'(\xi+,\xi)-g'(\xi-,\xi) = -\dfrac{1}{k_1}; \\ g(0-,\xi)=g(0+,\xi), \qquad k_1 g'(0-,\xi)=k_2 g'(0+,\xi). \end{cases}$$

If we solve the homogeneous equation in each of the three intervals and take into account the boundary conditions at $x=\pm 1$, we are left with four constants to be determined by the two interface conditions and the two matching conditions at the source. It is often preferable to begin with a solution which already satisfies the source conditions; an obvious candidate is the causal Green's function $h(x,\xi)=H(x-\xi)(\xi-x)/k_1$ for a rod of conductivity k_1. The desired Green's function $g(x,\xi)$ differs from h by a solution of the homogeneous equation in $-1<x<0$; in $0<x<1$, $g(x,\xi)$ is a solution of the homogeneous equation. In view of the boundary conditions we can write

$$(4.42)\quad g = \begin{cases} h(x,\xi)+A, & x<0, \\ B(1-x), & x>0. \end{cases}$$

It remains to apply the interface conditions to obtain

$$\frac{\xi}{k_1}+A=B, \qquad -1=k_2B,$$

so that

$$B=\frac{1}{k_2}, \qquad A=\frac{1}{k_2}-\frac{\xi}{k_1}.$$

We leave as an exercise the calculation of $g(x,\xi)$ when the source is in the right half of the rod.

What are the natural interface conditions for more complicated differential operators? We now present a method which avoids the limiting process described earlier. Suppose we want to solve the steady neutron diffusion problem

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}\right)+q(x)u=f(x), \tag{4.43}$$

where the source density f is a given piecewise continuous function, $k(x)$ is the diffusion coefficient, and $q(x)$ is related to the collision cross section and the multiplication factor [see (5.23), Chapter 0]. Normally one assumes that q is continuous and k smooth and then searches for a classical solution u. However, the left side $-(ku')'+qu$ can be piecewise continuous under weaker conditions on k and q. Suppose, for instance, that k and q are only piecewise continuous; then $-(ku')'+qu$ will be piecewise continuous if (a) ku' is piecewise smooth so that $(ku')'$ is defined as a piecewise continuous function, and (b) qu is piecewise continuous. Now, if ku' is piecewise smooth, it is certainly continuous, and therefore u' is piecewise continuous, so that u is continuous, and the condition on qu is automatically satisfied. In applying these ideas to concrete problems, we usually solve (4.43) in the subintervals where both k and q are continuous; this leaves us with constants of integration that are explicitly found by applying interface or matching conditions at the ends of subintervals. Two conditions are needed at each interface x_i:

$$\Delta(ku')_i=0, \qquad \Delta u_i=0, \tag{4.44}$$

where ΔF_i is the jump in F at x_i, that is, $F(x_i+)-F(x_i-)$. If only q is discontinuous at an interface, the matching conditions are

$$\Delta u_i=0, \qquad \Delta u_i'=0. \tag{4.45}$$

In one-dimensional quantum mechanics, the Schrödinger equation has the form

(4.46) $$u''+[E-V(x)]u=0,$$

where E is a constant and $V(x)$ is the potential. Often V is only piecewise continuous (as in problems of a rectangular well or a rectangular potential barrier). One then solves for u in the various intervals of continuity of $V(x)$; at the points where V is discontinuous the matching conditions are (4.45).

For a more complicated problem such as

$$(r_2u'')''+(r_1u')'+r_0u=f,$$

where the coefficients may only be piecewise continuous, one writes the left side as

$$[(r_2u'')'+r_1u']'+r_0u.$$

To make this piecewise continuous, we need the following: (a) r_2u'' piecewise smooth (hence r_2u'' continuous, u'' piecewise continuous, u' and u continuous); (b) $(r_2u'')'+r_1u'$ piecewise smooth (hence continuous); (c) r_0u piecewise continuous [follows automatically from (a)]. This gives us the four interface conditions:

(4.47) $$\Delta u_i=0, \qquad \Delta u_i'=0, \qquad \Delta(r_2u'')_i=0, \qquad \Delta[(r_2u'')'+r_1u']_i=0.$$

As an illustration, consider the small transverse deflection of a beam of constant cross section whose stiffness changes abruptly at $x=0$. If $E(x)$ is the stiffness of the beam, the deflection satisfies

$$(E(x)u'')''=f(x),$$

so that, according to (4.47), the interface conditions at $x=0$ are

(4.48) $$\Delta u_0=0, \qquad \Delta u_0'=0, \qquad \Delta(Eu'')_0=0, \qquad \Delta(Eu'')_0'=0.$$

These conditions have a very simple interpretation: the deflection, slope, moment, and shear are all continuous at $x=0$. In particular, if E is the constant E_1 for $x<0$ and the constant E_2 for $x>0$, these conditions become

$$u(0+)=u(0-), \qquad u'(0+)=u'(0-),$$
$$E_2u''(0+)=E_1u''(0-), \qquad E_2u'''(0+)=E_1u'''(0-).$$

In Exercises 4.6 and 4.7 we give examples of different kinds of problems for composite beams that do *not* lead to natural interface conditions.

Exercises

4.1. Let q^2 be a positive constant. Find Green's function $g(x,\xi)$ satisfying

$$-g''+q^2g=\delta(x-\xi),\quad 0<x,\xi<1;\qquad g'(0,\xi)=g'(1,\xi)=0$$

by the direct method of Section 2, that is, by starting with two solutions u_1, u_2 of the homogeneous equation satisfying, respectively, the end conditions at $x=0$ and $x=1$ and then matching them under the load. Compare your result with the one obtained by images, (4.17). Do you notice anything strange as $q \to 0$?

4.2. (a) Show that the electrostatic potential for a line source of uniform unit density is $u=(1/2\pi)\log(1/\rho)$, where ρ is the cylindrical coordinate measured from the line source (which coincides with the z axis).

(b) Consider the two-dimensional problem

$$-\Delta u=-\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}=0,\quad x_2>0,\ -\infty<x_1<\infty; \tag{4.49}$$
$$u(x_1,0)=h(x_1),$$

where $h(x_1)$ is a given function. First find Green's function $g(x,\xi)$ for 0 boundary data by the method of images [using part (a)]. Then write the solution of (4.49) by using (1.5) as

$$u(x_1,x_2)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x_2}{x_2^2+(x_1-\xi_1)^2}h(\xi_1)\,d\xi_1. \tag{4.50}$$

4.3. An elastic beam is subject to a restoring force proportional to the local displacement and tending to oppose it. If a transverse distributed load $f(x)$ is applied, the appropriate differential equation satisfied by the deflection $u(x)$ is

$$\frac{d^4u}{dx^4}+k^4u=f(x),$$

where k^4 is a positive constant regarded as known.

(a) Find the deflection in an infinite beam $-\infty < x < \infty$, when the applied load is a unit concentrated load at $x=\xi$. This deflection is the free space Green's function.
(b) Find the causal Green's function for the problem.
(c) For a beam simply supported at its ends $x=0$ and $x=1$, find Green's function first by using the causal Green's function of part (b) and then by using the method of images.

4.4. Consider neutron diffusion in all of three-dimensional space. The ball $|x|<R$ and its exterior are different homogeneous media that are in perfect contact. A uniform layer of sources of surface density ρ is located on the sphere $|x|=a$, where $a<R$. Set up and solve the boundary value problem for the neutron density. Consider also the limiting case of a point source at the origin.

4.5. *Quasi-derivatives.* Consider the equation of order $2p$:

$$(4.51)\qquad (r_p(x)u^{(p)})^{(p)} + \cdots + (r_1u')' + r_0u = f.$$

We can write the left side as

$$u^{[2p]}(x),$$

where the *quasi-derivatives* $u^{[k]}(x)$ are defined by

$$u^{[0]}=u, \qquad u^{[1]}=u', \dots, u^{[p-1]}=u^{(p-1)},$$
$$u^{[p]}=r_pu^{(p)}, \qquad u^{[p+1]}=(u^{[p]})'+r_{p-1}u^{[p-1]},$$
$$u^{[p+2]}=(u^{[p+1]})'+r_{p-2}u^{[p-2]}, \dots .$$

Show that the differential equation (4.51) then becomes

$$u^{[2p]}=f,$$

and that the natural interface conditions take the simple form that $u, u^{[1]}, \dots, u^{[2p-1]}$ be continuous.

4.6. A simply supported composite beam of constant cross section occupies the interval $0<x<2l$. The left half has $EI=1$, whereas the right half is *rigid* and the two halves are welded together at $x=l$. A concentrated unit transverse force is applied at $x=\xi$, where $0<\xi<l$. Draw the shear, moment, slope, and deflection diagrams. Express these analytically.

4.7. A homogeneous beam of constant cross section is attached to a string. The beam and string are stretched under tension H between the fixed points $x=0$ and $x=2l$, the beam occupying the interval $0<x<l$ and the string the interval $l<x<2l$. The left end of the beam is simply supported. A transverse concentrated unit force is applied at $x=l/2$. What are the interface conditions at $x=l$? Find the deflection.

4.8. Consider the case of a steep potential well or barrier in (4.46), which can be ideally represented by a potential $V(x)=\alpha\delta(x)$, where α is a real number. Although such a problem does not fall into the class studied in Section 4, it can nevertheless be solved. The principal interest is in the matching (connection) conditions at $x=0$. If we assume u continuous at the origin, a formal integration of (4.46) gives $u'(0+)-u'(0-)=\alpha u(0)$. Show that the same result is obtained by replacing the delta function in (4.46) by the sequence

$$f_n(x)=\begin{cases} n, & 0<x<\dfrac{1}{n}, \\ 0, & \text{otherwise}, \end{cases}$$

and then proceeding to the limit as $n\to\infty$.

REFERENCE AND ADDITIONAL READING

Protter, M. H. and Weinberger, H., *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N. J., 1967.

2
The Theory of Distributions

1. BASIC IDEAS, DEFINITIONS, EXAMPLES

Various examples of Chapter 1, Section 4 show that one frequently encounters sources that are nearly instantaneous (if time is the independent variable) or almost localized (if a space coordinate is the independent variable). To avoid the cumbersome study of the detailed functional dependence of such sources, we would like to replace them by idealized sources which are truly instantaneous or localized; such idealized sources are said to be *impulsive* or *concentrated* (as opposed to distributed sources). Typical instances of such sources are the concentrated forces and moments of solid mechanics; the heat sources and dipoles in heat conduction; the point masses in the theory of the gravitational potential; the impulsive forces in acoustics and in impact mechanics; the fluid sources and vortices of incompressible fluid mechanics; and the point charges, dipoles, multipoles, line charges, and surface layers in electrostatics.

What do we expect from a mathematical theory of concentrated sources? First, there should be a clear and unambiguous mathematical framework in which such sources have equal standing with distributed sources. Second, a method should be provided for calculating the response to a concentrated source, that is, a means of interpreting and solving a differential equation whose inhomogeneous term is a concentrated source. Third, if a concentrated source is "approximated" by a sequence of distributed sources, the response to the concentrated source should be a suitable limit of the sequence of responses to the distributed sources.

Functions as Linear Functionals

Consider a real-valued, continuous function on R_n. The function f is a rule which associates with each point x in R_n a real number $y=f(x)$, the value of f at x.