Math 701 Final Version A

1. Answer one of the following questions:

(i) For $x \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n\times n}$ prove that $\|A\|_2 = \rho(A^T A)^{1/2}$.

(ii) Prove every nonconstant polynomial has at least one root in $\mathbf{C}$.

2. Answer one of the following questions:

(i) Suppose $A \in \mathbf{R}^{n\times n}$ is symmetric and positive definite. Prove that the problem of solving $Ax = b$ is equivalent to the problem of minimizing the quadratic form $q(x) = \langle x, Ax\rangle - 2\langle x, b\rangle$.

(ii) Let f'' be continuous and let r be a simple zero of f. Prove there is a neighborhood of r and a constant C such that if Newton's method is started in that neighborhood, the successive points become steadily closer to r and satisfy $|x_{n+1} - r| \leq C(x_n - r)^2$ for $n \in \mathbf{N}$.

3. Let $A \in \mathbf{R}^{n\times n}$ and $b, x \in \mathbf{R}^n$ such that $Ax = b$. Given $\tilde{x} \in \mathbf{R}^n$ prove that

$$\frac{1}{\kappa(A)} \frac{\|b - A\tilde{x}\|}{\|b\|} \leq \frac{\|x - \tilde{x}\|}{\|x\|} \leq \kappa(A) \frac{\|b - A\tilde{x}\|}{\|b\|}$$

where $\kappa(A) = \|A\|\|A^{-1}\|$.

4. Prove one of the following lemmata:

Lemma 1. The matrix $I - vv^*$ is unitary if and only if $\|v\|_2^2 = 2$ or $v = 0$.

Lemma 2. Let x and y be two vectors such that $\|x\|_2 = \|y\|_2$ and $\langle x, y \rangle$ is real. Then there exists a unitary matrix U of the form $I - vv^*$ such that $Ux = y$

5. Let f be a function in $C^{n+1}[a,b]$ and let p be the polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in the interval $[a,b]$. Prove that to each x in $[a,b]$ there corresponds a point ξ_x in (a,b) such that

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i).$$

6. Describe the Gram Schmidt algorithm and explain how it can be used to find a family of polynomials $\{\, p_k : k = 0, 1, \ldots, n \,\}$ where each p_k has degree less than or equal to k and

$$\int_0^1 p_i(x) p_j(x)(x^2+1)\,dx = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

7. Let $f\colon \mathbf{R} \to \mathbf{R}$ and $x_i \in \mathbf{R}$ for $i = 0, \ldots, k$. Consider the polynomial p_k given by

$$p_k(x) = \sum_{i=0}^{k} c_i \prod_{j=0}^{i-1} (x - x_j)$$

where c_i have been chosen so $p_k(x_i) = f(x_i)$ for $i = 0, \ldots, k$. Explain how to use Newton's divided difference formula to find the c_i.

8. Define what it means for a matrix to be positive definite.

9. Define a Haar subspace.

10. Describe how to solve the equation $ax^2 + bx + c = 0$ using the quadratic formula in a way that minimizes rounding error and loss of precision.