Conjugate Gradient Method

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output. If you choose to work in a group of two, please turn in independently prepared reports.

1. The $n \times n$ Hilbert matrix H is defined as the matrix with entries

$$h_{ij} = \frac{1}{i+j-1} \qquad \text{where} \qquad i,j = 1,2,\ldots,n.$$

Prove for any $n \in \mathbf{N}$ that H is symmetric and positive definite.

2. Let w be a vector of length n given by

$$w_i = \frac{1}{3} \qquad \text{where} \qquad i = 1,2,\ldots,n.$$

Write a subroutine that computes Hw without storing the entire matrix H into memory. Write a program to find $w \cdot Hw$ for $n = 10^k$ where $k = 1,2,3,4$. The output from your program should look something like

```
k      n                   w.Hw
1     10  1.486158673723173e+00
2    100  1.534785401070720e+01
3   1000  1.539771651244306e+02
4  10000  1.540271513744324e+03
```

3. Write a program that solves $Hx = b$ by the conjugate gradient method. Test your program for $n = 1000$ by choosing $b = Hw$ where w is the vector given above and $x = 0$ as the initial guess for x. The output should look sometime like

```
  k                  |w-x|                  |b-Ax|
  0  1.054092553389460e+01   1.698808462143129e+01
  1  5.489846287694768e+00   4.406748857997090e+00
  2  3.456551995574104e+00   1.170268545385278e+00
  3  2.122967238170827e+00   2.607637508249531e-01
...                    ...                     ...
 50  3.406923944634312e-04   1.256784027692001e-10
```

4. What happens when one tries to use the conjugate gradient method to solve $Ax = b$ when A is not symmetric or positive definite?

5. Let A be an invertible matrix. Prove $B = A^T A$ is symmetric and positive definite.

6. Consider the following method for solving $Ax = b$ when A is an arbitrary invertible matrix. Multiply both sides by A^T to obtain

$$Bx = c \qquad \text{where} \qquad B = A^T A \quad \text{and} \quad c = A^T b.$$

Then solve $Bx = c$ using the conjugate gradient method. Test this method on some interesting matrices A which are neither symmetric nor positive definite.