Math 701 Quiz 2 Version A

1. Find a suitable trigonometric identity so that $1 - \cos x$ can be accurately computed for small x with calls to the system functions for $\sin x$ or $\cos x$.

Recall the sine angle addition formula

$$\sin(a + b) = \sin a \cos b + \cos a \sin b.$$

This formula is easy to remember because if its symmetry. Differentiate with respect to a to obtain the cosine angle addition formula

$$\cos(a + b) = \cos a \cos b - \sin a \sin b.$$

Now, setting $a = x/2$ and $b = x/2$ yields the half-angle formula

$$\cos x = \cos^2(x/2) - \sin^2(x/2).$$

Subtracting the above identity from the Pythagorean theorem $1 = \cos^2(x/2) + \sin^2(x/2)$ results in the trigonometric identity $1 - \cos x = 2\sin^2(x/2)$ which is suitable to accurately approximate $1 - \cos x$ for small values.

2. State Taylor's theorem including all hypothesis and the remainder term.

Taylor's Theorem. Let $f\colon \mathbf{R} \to \mathbf{R}$ be an $n+1$ times continuously differentiable function. Then

$$f(x + h) = \sum_{k=0}^{n} \frac{h^n}{n!} f^{(n)}(x) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for some ξ between x and $x + h$.

3. State the power method for finding the largest eigenvalue and corresponding eigenvector of a symmetric positive semidefinite matrix $A \in \mathbf{R}^{d\times d}$.

Let $x_0 \in \mathbf{R}^d$ be chosen randomly and define

$$y_n = Ax_n \qquad \text{and} \qquad x_{n+1} = y_n/\|y_n\| \qquad \text{for} \qquad n = 0, 1, 2, \ldots.$$

Then

$$\|y_n\| \to \lambda \qquad \text{and} \qquad x_n \to \xi \qquad \text{as} \qquad n \to \infty$$

where λ is the largest eigenvalue of A and ξ is its corresponding eigenvector.

4. Let f be a twice continuously differentiable function and p be a point such that $f(p) = 0$ and $f'(p) \neq 0$. Prove that Newton's method $x_{n+1} = x_n - f(x_n)/f'(x_n)$ is quadratically convergent provided x_0 is close enough to p.

Since $f'(p) \neq 0$ there exists $\delta > 0$ such that

$$A = \min\{\, |f'(t)| : t \in [p-\delta, p+\delta]\,\} > 0.$$

Further define

$$B = \max\{\, |f''(t)| : t \in [p-\delta, p+\delta]\,\}.$$

Now let $\epsilon = \min\{\delta, M^{-1}\}$ where $M = B/(2A)$. We claim the condition $|x_0 - p| < \epsilon$ is sufficient to guarantee

$$\lim_{n\to\infty} x_n = p \qquad \text{and} \qquad |x_{n+1} - p| \le M|x_n - p|^2 \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Define $e_n = x_n - p$. By Taylor's theorem there is ξ_n between p and x_n such that

$$0 = f(p) = f(x_n) + (p - x_n)f'(x_n) + \frac{1}{2}(p - x_n)^2 f''(\xi_n).$$

It follows that

$$0 = \frac{f(x_n)}{f'(x_n)} - e_n + \frac{1}{2}e_n^2\frac{f''(\xi_n)}{f'(x_n)} \quad \text{or equivalently} \quad e_n - \frac{f(x_n)}{f'(x_n)} = \frac{1}{2}e_n^2\frac{f''(\xi_n)}{f'(x_n)}.$$

Consequently

$$|e_{n+1}| = |x_{n+1} - p| = \left|x_n - \frac{f(x_n)}{f'(x_n)} - p\right| = \left|e_n - \frac{f(x_n)}{f'(x_n)}\right| = \frac{1}{2}|e_n|^2\left|\frac{f''(\xi_n)}{f'(x_n)}\right|.$$

Suppose for induction that $|x_n - p| < \epsilon$ as is the case when $n = 0$. Then $\epsilon \le \delta$ implies

$$|e_{n+1}| = \frac{1}{2}|e_n|^2\left|\frac{f''(\xi_n)}{f'(x_n)}\right| \le \frac{1}{2}\frac{\max\{\,|f''(t)| : t \in [p-\epsilon, p+\epsilon]\,\}}{\min\{\,|f'(t)| : t \in [p-\epsilon, p+\epsilon]\,\}} \le M|e_n|^2.$$

Since $M|e_n| \le M\epsilon \le 1$ then $|e_{n+1}| \le |e_n|$ which implies $|x_{n+1} - p| < \epsilon$ and completes the induction. In particular, we have shown that

$$|e_{n+1}| \le M|e_n|^2 \qquad \text{and} \qquad |e_{n+1}| \le |e_n| \qquad \text{for} \qquad n = 0, 1, 2, \ldots.$$

It remains to show $x_n \to p$ as $n \to \infty$. The second inequality above immediately implies $|e_n| \le |e_0|$. Define $\gamma = M|e_0|$. Since $M|e_0| < M\epsilon \le MM^{-1} = 1$ then $\gamma < 1$. Now

$$|e_{n+1}| \le M|e_n|^2 \le (M|e_0|)|e_n| = \gamma|e_n|$$

implies $|e_n| \le \gamma^n|e_0|$ for all n. Since $\gamma^n \to 0$ as $n \to \infty$ it follows that $x_n \to p$.

5. Let $B \in \mathbf{R}^{d\times d}$ and consider the matrix norm given by

$$\|B\|_2 = \max\{\, \|Bx\|_2 : \|x\|_2 = 1\,\} \qquad \text{where} \qquad \|x\|_2 = \Big(\sum_{i=1}^{d} |x_i|^2\Big)^{1/2}.$$

Prove $\|B\|_2 = \rho(B^TB)^{1/2}$ where $\rho(A) = \max\{\,|\lambda| : \lambda \text{ is an eigenvalue of } A\,\}$.

Let $A = B^TB$. Then $A \in R^{d\times d}$ is a symmetric positive semidefinite matrix. The spectral theorem for real symmetric matrices implies that there exists an orthonormal basis of eigenvectors ξ_i with corresponding eigenvalues λ_i for $i = 1, 2, \ldots, d$ such that

$$A\xi_i = \lambda_i\xi_i \qquad \text{and} \qquad \xi_i \cdot \xi_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Since A is semidefinite it is further the case that $\lambda_i \ge 0$ for all i. Given $x \in \mathbf{R}^d$ there exists constants $c_i \in \mathbf{R}$ such that

$$x = \sum_{k=1}^{d} c_k\xi_k.$$

Consequently, the orthonormality of the ξ_k's implies

$$\|x\|_2^2 = x\cdot x = \sum_{k=1}^{d} c_k\xi_k \cdot \sum_{\ell=1}^{d} c_\ell\xi_\ell = \sum_{k=1}^{d}\sum_{\ell=1}^{d} c_kc_\ell\xi_k\cdot\xi_\ell = \sum_{k=1}^{d} c_k^2.$$

Similarly

$$\|Bx\|_2^2 = Bx\cdot Bx = x\cdot Ax = \sum_{k=1}^{d} c_k\xi_k \cdot \sum_{\ell=1}^{d} c_\ell\lambda_\ell\xi_\ell = \sum_{k=1}^{d} \lambda_k c_k^2.$$

Since $\lambda_k \ge 0$ then $\lambda_k = |\lambda_k|$ and it follows that

$$\|B\|_2^2 = \max\Big\{\, \sum_{k=1}^{d} \lambda_k c_k^2 : \sum_{k=1}^{d} c_k^2 = 1\,\Big\} = \max\Big\{\, \sum_{k=1}^{d} |\lambda_k| c_k^2 : \sum_{k=1}^{d} c_k^2 = 1\,\Big\}.$$

The above maximum may be interpreted as the maximum over all possible weighted averages of the $|\lambda_k|$'s. Since any weighted average is between the smallest and largest, then further placing all the weight on the largest $|\lambda_k|$ yields that

$$\|B\|_2^2 = \max\{\,|\lambda_k| : k = 1, 2, \ldots, d\,\}$$

or equivalently that $\|B\|_2 = \rho(B^TB)^{1/2}$.

6. Consider the matrix $A \in \mathbf{R}^{4\times 4}$ given by

$$A = \begin{bmatrix} -8 & -6 & -6 & -1 \\ 4 & 8 & -4 & 1 \\ -4 & -10 & 2 & -8 \\ 3 & -7 & 9 & 7 \end{bmatrix}$$

(i) Find $\|A\|_1$.

Since

$$\begin{aligned} \textstyle\sum_{j=1}^{d} |a_{1,j}| &= 8+4+4+3 = 19 \\ \textstyle\sum_{j=1}^{d} |a_{2,j}| &= 6+8+10+7 = 31 \\ \textstyle\sum_{j=1}^{d} |a_{3,j}| &= 6+4+2+9 = 21 \\ \textstyle\sum_{j=1}^{d} |a_{4,j}| &= 1+1+8+7 = 17 \end{aligned}$$

then

$$\|A\|_1 = \max\Big\{ \sum_{j=1}^{d} |a_{ij}| : i = 1, \ldots d \Big\} = \max\{19, 31, 21, 17\} = 31.$$

(ii) Find $\|A\|_\infty$.

Since

$$\begin{aligned} \textstyle\sum_{i=1}^{d} |a_{i,1}| &= 8+6+6+1 = 21 \\ \textstyle\sum_{i=1}^{d} |a_{i,2}| &= 4+8+4+1 = 17 \\ \textstyle\sum_{i=1}^{d} |a_{i,3}| &= 4+10+2+8 = 24 \\ \textstyle\sum_{i=1}^{d} |a_{i,4}| &= 3+7+9+7 = 26 \end{aligned}$$

then

$$\|A\|_1 = \max\Big\{ \sum_{i=1}^{d} |a_{ij}| : j = 1, \ldots d \Big\} = \max\{21, 17, 24, 26\} = 26.$$

7. Prove or disprove whether $\|A^2\|_\infty = \|A\|_\infty^2$ holds in general for matrices $A \in \mathbf{R}^{4\times 4}$.

This is false. While the matrix A defined above could demonstrate that $\|A^2\|_\infty \neq \|A\|_\infty^2$, an easier example is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{for which} \qquad A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case $\|A\|_\infty^2 = 1$ and $\|A^2\|_\infty = 0$.