Math 702 Exam 1 Version A

1. State Taylor's theorem with remainder term.

2. Give the definition of truncation error and consistency.

3. Give the definition of stability.

4. Prove one of the following:

(i) Theorem 2.5.2 Lax Theorem. If a two-level difference scheme

$$u^{n+1} = Qu^n + \Delta t G^n$$

is accurate of order (p, q) in the norm $\|\cdot\|$ to a well-posed linear initial-value problem and is stable with respect to the norm $\|\cdot\|$, then it is convergent of order (p, q) with respect to the norm $\|\cdot\|$.

(ii) Convergence of Euler's Central Difference Method for the Heat Equation. Consider the partial differential equation

$$\begin{aligned} &v_t = \nu v_{xx} \qquad \text{for } x \in (0,1) \text{ and } t \in (0,T) \\ &v(0,t) = v(1,t) = 0 \\ &v(x,0) = f(x) \end{aligned}$$

and the finite difference scheme

$$\begin{aligned} &u_k^{n+1} = u_k^n + \nu \frac{\Delta t}{\Delta x^2}\big(u_{k+1}^n - 2u_k^n + u_{k-1}^n\big) \qquad \text{for } k = 1, \ldots, K-1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 0, \ldots, N \\ &u_0^n = u_K^n = 0 \\ &u_k^0 = f(k\Delta x) \end{aligned}$$

where $\Delta x = 1/K$ and $\Delta t = T/N$. Find conditions that guarantee the finite difference scheme converges to the solution of the differential equation and then prove the convergence.

5. Work one of the exercises from section 2.3 in our book that appear on the following two pages.

consistency result and discuss briefly the conditions necessary to obtain norm consistency (i.e. bounded derivatives for the sup-norm, $\ell_{2,\Delta x}$ bounds on the derivatives for the $\ell_{2,\Delta x}$-norm and uniform boundedness of Q_1^{-1} for implicit schemes. And, of course, if the scheme cannot be shown to be either pointwise or norm consistent, this explanation should be included.

HW 2.3.1 Determine the order of accuracy of the following difference equations to the given initial–value problems.
(a) Explicit scheme for heat equation with lower order term (FTCS).

$$u_k^{n+1} = u_k^n - \frac{a\Delta t}{2\Delta x}\delta_0 u_k^n + \frac{\nu\Delta t}{\Delta x^2}\delta^2 u_k^n$$

$$v_t + av_x = \nu v_{xx}$$

(b) Implicit scheme for heat equation with lower order term (BTCS).

$$u_k^{n+1} + \frac{a\Delta t}{2\Delta x}\delta_0 u_k^{n+1} - \frac{\nu\Delta t}{\Delta x^2}\delta^2 u_k^{n+1} = u_k^n$$

$$v_t + av_x = \nu v_{xx}$$

(c) Crank-Nicolson Scheme

$$u_k^{n+1} - \frac{\nu\Delta t}{2\Delta x^2}\delta^2 u_k^{n+1} = u_k^n + \frac{\nu\Delta t}{2\Delta x^2}\delta^2 u_k^n$$

$$v_t = \nu v_{xx}$$

Explain why it is logical to consider the consistency of this scheme at the point $(k\Delta x, (n+1/2)\Delta t)$ rather than at $(k\Delta x, n\Delta t)$ or $(k\Delta x, (n+1)\Delta t)$.
(d) Dufort-Frankel Scheme

$$u_k^{n+1} = \frac{2r}{1+2r}(u_{k+1}^n + u_{k-1}^n) + \frac{1-2r}{1+2r}u_k^{n-1}$$

where $r = \frac{\Delta t}{\Delta x^2}$.

$$v_t = v_{xx}$$

Is the any logical condition that can be placed on this scheme that will make it consistent?
(e) Forward-time, forward-space for a hyperbolic equation (FTFS).

$$u_k^{n+1} = u_k^n - \frac{a\Delta t}{\Delta x}(u_{k+1}^n - u_k^n)$$

$$v_t + av_x = 0$$

HW 2.3.2 (a) Show that the following difference scheme is a $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^4)$ approximation of $v_t = \nu v_{xx}$ (where $r = \nu\Delta t/\Delta x^2$).

$$u_k^{n+1} = u_k^n + r\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right)$$

Discuss the assumptions that must be made on the derivatives of the solution to the partial differential equation that are necessary to make the above statement true.

(b) Show that the following difference scheme is a $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^4)$ approximation of $v_t + av_x = 0$ (where $R = a\Delta t/\Delta x$).

$$u_k^{n+1} = u_k^n - \frac{R}{2}\delta_0 u_k^n + \frac{R}{12}\delta^2\delta_0 u_k^n + \frac{R^2}{2}\left(\frac{4}{3} + R^2\right)\delta^2 u_k^n$$
$$-\frac{R^2}{8}\left(\frac{1}{3} + R^2\right)\delta_0^2 u_k^n, \quad k = 1, 2, \cdots$$

Discuss the assumptions necessary for the above statement to be true.

HW 2.3.3 Determine the order of accuracy of the following difference equations to the partial differential equation

$$v_t + av_x = 0.$$

(a) Leapfrog scheme $u_k^{n+1} = u_k^{n-1} - R\delta_0 u_k^n$

(b) $u_k^{n+1} = u_k^{n-1} - R\delta_0 u_k^n + \frac{R}{6}\delta^2\delta_0 u_k^n$

(c) $u_k^{n+1} = u_k^{n-1} - R\delta_0 u_k^n + \frac{R}{6}\delta^2\delta_0 u_k^n - \frac{R}{30}\delta^4\delta_0 u_k^n$ where $\delta^4 = \delta^2\delta^2$.

(d) $u_k^{n+2} = u_k^{n-2} - \frac{2R}{3}\left(1 - \frac{1}{6}\delta^2\right)\delta_0\left(2u_k^{n+1} - u_k^n + 2u_k^{n-1}\right)$

2.3.2 Initial–Boundary–Value Problems

Just as convergence for initial–boundary–value problems had to be treated differently from initial-value problems, we must also be careful when considering the consistency for initial–boundary–value problems. Pointwise consistency will be the same as it was in Definition 2.3.1 except that we must now also analyze any boundary conditions that contain an approximation. This is explained below.

For norm consistency, as we did in Section 2.2.2, we consider a sequence of partitions of the interval $[0,1]$ defined by a sequences of spatial increments $\{\Delta x_j\}$ and a sequence of the appropriate spaces, $\{X_j\}$, with norms $\{\|\cdot\|_j\}$. Then *Definitions 2.3.2 and 2.3.3 carry over to the initial–boundary–value problem* by replacing the norm in equations (2.3.5) and (2.3.6) by the sequence of norms $\|\cdot\|_j$. The important difference between the initial–value problem and initial–boundary–value problem is in writing the initial–boundary–value problem in some form that can be analyzed (say in a form like equation (2.3.4) or (2.3.16)) and set in some logical sequence of vector spaces. For example, it is easy to see that if we consider the explicit scheme (2.2.2) for an initial–boundary–value problem with Dirichlet