Nonlinear Schrödinger Equation

Your work should be presented in the form of a typed report using clear and properly punctuated English. Where appropriate include full program listings and output. If you choose to work in a group of two, please turn in independently prepared reports.

1. Consider the nonlinear Schrödinger equation

$$\begin{cases} iv_t = v_{xx} + 2|v|^2 v & \text{for } (x,t) \in \mathbf{R} \times (0,T) \\ v(x,0) = f(x) & \text{for } x \in \mathbf{R}. \end{cases}$$

   Suppose for some $L > 0$ that $f(x) = f(x+L)$ for $x \in \mathbf{R}$. Prove the resulting solution v satisfies $v(x,t) = v(x+L,t)$ for all $(x,t) \in \mathbf{R} \times (0,T)$. We shall say that v satisfies the nonlinear Schrödinger equation with L-periodic boundary conditions.

2. Let v be a solution to the nonlinear Schrödinger equation with L-periodic boundary conditions. Let $\Delta x = L/K$ and $\Delta t = T/N$ and consider the finite difference method for approximating v on $[0,L] \times [0,T]$ given by

$$\begin{cases} u_k^{n+1} = u_k^{n-1} - \dfrac{2i\Delta t}{\Delta x^2}\delta^2 u_k^n - 4i\Delta t|u_k^n|^2 u_k^n & \text{for } n = 1,\ldots,N-1 \\ & \text{and } k = 1,\ldots,K \\ u_k^1 = u_k^0 - \dfrac{i\Delta t}{\Delta x^2}\delta^2 u_k^0 - 2i\Delta t|u_k^0|^2 u_k^0 & \text{for } k = 1,\ldots,K \\ u_0^n = u_K^n \quad \text{and} \quad u_{K+1}^n = u_1^n & \text{for } n = 0,\ldots,N \\ u_k^0 = f(k\Delta x) & \text{for } k = 1,\ldots,K. \end{cases}$$

   For $L = 10$ and $T = 0.2$ choose K and N sufficiently large to approximate $v(5, 0.2)$ to at least 3 significant digits for the personalized initial condition f given below

$$\begin{aligned} f_{\text{Alexander}}(x) &= 2\exp\big(3i\sin(\omega x)\big) + \sin(3\omega x) \\ f_{\text{Anthony}}(x) &= \exp(i\omega x) - \sin(2\omega x) + \exp(5i\omega x) \\ f_{\text{Brian}}(x) &= 0.5\exp(i\omega x) + 0.5\sin(2\omega x) - 2i\cos(3\omega x) \\ f_{\text{Jordan}}(x) &= 2\exp(-i\omega x) - i\cos(3\omega x) \\ f_{\text{Joseph}}(x) &= 0.5\exp(i\omega x) - \exp(2i\omega x) + i\exp(3i\omega x) + 0.5\exp(5i\omega x) \\ f_{\text{Kyle}}(x) &= i\sin(\omega x) + \cos(2\omega x) + \exp(-3i\omega x) \\ f_{\text{Masakazu}}(x) &= 1 + \exp(i\omega x) \\ f_{\text{Sarah}}(x) &= \exp(i\omega x) + \exp(-2i\omega x) + \exp(3i\omega x) \\ f_{\text{Shijie}}(x) &= \sin\big(3\cos(\omega x)\big) + 2\exp(2i\omega x) \end{aligned}$$

   where $\omega = \pi/5$.

3. Approximate $v(5, 0.2)$ using one of the other methods found, for example, in

   Thiab R. Taha, Mark J. Ablowitz, Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations II: Numerical, Nonlinear Schrödinger Equation, *J. Comput. Phys.*, **55** (1984), pp. 203–230.

   Please include a bibliographic reference to the method you implemented as a comment in your source code.