$\boxed{\text{Chapter } 2.3}$ HW 2.3.1(a)(d)(g), HW 2.3.2 (a), HW 2.3.3 (a)(b) $\boxed{\text{MATH } 702}$

In order to reduce writing I'll adopt the notation that when a function is written with no arguments it is assumed to be evaluated at the point (x_k, t_n). Thus Taylor's expansion of $V^n_{k+1} = V(x_{k+1}, t_n)$ can be written as

$$V^n_{k+1} = V + \Delta x V_x + \frac{\Delta x^2}{2} V_{xx} + \frac{\Delta x^3}{3!} V_{xxx} + \frac{\Delta x^4}{4!} V_{xxxx} + O(\Delta x^5)$$

To further simplify the writing $V_{(n)}$ shall be used to denote the n-th partial derivative of V with respect to x. Thus

$$V^n_{k+1} = V + \Delta x V_x + \frac{\Delta x^2}{2} V_{xx} + \frac{\Delta x^3}{3!} V_{(3)} + \frac{\Delta x^4}{4!} V_{(4)} + O(\Delta x^5)$$

We start with some formulas that result from Taylor's theorem

$$\delta_0 V^n_k = 2\Delta x V_x + \frac{\Delta x^3}{3} V_{(3)} + \frac{\Delta x^5}{60} V_{(5)} + O(\Delta x^7)$$

$$\delta^2 V^n_k = \Delta x^2 V_{xx} + \frac{\Delta x^4}{12} V_{(4)} + O(\Delta x^6)$$

$$\delta_0^2 V_k^n = 4\Delta x^2 V_{xx} + \frac{4}{3}\Delta x^4 V_{(4)} + O(\Delta x^6)$$

$$\delta^2 \delta_0 V_k^n = 2\Delta x^3 V_{(3)} + \frac{1}{2}\Delta x^5 V_{(5)} + O(\Delta x^7).$$

$$V_{k+1}^n + V_{k-1}^n \approx 2V + \Delta x^2 V_{(2)} + \frac{1}{12}\Delta x^4 V_{(4)} + O(\Delta x^6)$$

$$V_{k+2}^n + V_{k-2}^n = 2V + 4\Delta x^2 V_{(2)} + \frac{4}{3}\Delta x^4 V_{(4)} + O(\Delta x^6),$$

HW 2.3.1(a) $\quad v_t + a v_x = \nu v_{xx}$

$$v_k^{n+1} = v_k^n - \frac{a\Delta t}{2\Delta x}\delta_0 v_k^n + \frac{\nu\Delta t}{\Delta x^2}\delta^2 v_k^n + \Delta t\,\tau^n$$

Thus

$$\cancel{v_k^n} + \Delta t\, v_t + O(\Delta t^2) = \cancel{v_k^n}$$
$$- \frac{a\Delta t}{2\Delta x}\left(2\Delta x\, v_x + \frac{\Delta x^3}{3} v_{(3)} + O(\Delta x^5)\right)$$
$$+ \frac{\nu\Delta t}{\Delta x^2}\left(\Delta x^2 v_{xx} + \frac{\Delta x^4}{12} v_{(4)} + O(\Delta x^6)\right) + \Delta t\,\tau^n$$

Gives

$$\cancel{v_t} + O(\Delta t) = -a\left(\cancel{v_x} + \frac{\Delta x^2}{6} v_{(3)} + O(\Delta x^4)\right) +$$
$$+ \nu\left(\cancel{v_{xx}} + \frac{\Delta x^2}{12} v_{(4)} + O(\Delta x^4)\right) + \tau^n$$

So

$$\tau^n = a\frac{\Delta x^2}{6} v_{(3)} - \nu\frac{\Delta x^2}{12} v_{(4)} + O(\Delta t) + O(\Delta x^4) = O(\Delta t) + O(\Delta x^4).$$

HW 2.3.1 (d)

$$V_k^{n+1} = \frac{2r}{1+2r}\left(V_{k+1}^n + V_{k-1}^n\right) + \frac{1-2r}{1+2r} V_k^{n-1} + \Delta t\, \tau^n, \qquad V_t = V_{xx}, \qquad r = \frac{\Delta t}{\Delta x^2}$$

Expanding gives

$$V_{k+1}^n + V_{k-1}^n = V + \cancel{\Delta x V_x} + \frac{\Delta x^2}{2} V_{xx} + \cancel{\frac{\Delta x^3}{3!} V_{(3)}} + \frac{\Delta x^4}{4!} V_{(4)} + \cancel{\frac{\Delta x^5}{5!} V_{(5)}} + O(\Delta x^6)$$
$$+ V - \cancel{\ } + \cancel{\ } - \cancel{\ } + \cancel{\ } - \cancel{\ } + O(\Delta x^6)$$
$$= 2V + \Delta x^2 V_{xx} + \frac{1}{12}\Delta x^4 V_{(4)} + O(\Delta x^6)$$

$$\cancel{V} + \Delta t V_t + \frac{\Delta t^2}{2} V_{tt} + \frac{\Delta t^3}{6} V_{ttt} + O(\Delta t^4) = \frac{2r}{1+2r}\left(\cancel{2V} + \Delta x^2 V_{xx} + \frac{1}{12}\Delta x^4 V_{(4)} + O(\Delta x^6)\right)$$
$$+ \frac{1-2r}{1+2r}\left(\cancel{V} - \Delta t V_t + \frac{\Delta t^2}{2} V_{tt} - \frac{\Delta t^3}{6} V_{ttt} + O(\Delta t^4)\right) + \cancel{\Delta t}\,\tau^n$$

Therefore

$$\cancel{\frac{1+2r}{1+2r} V_t} + \cancel{\frac{\Delta t}{2} V_{tt}} + \boxed{\frac{\Delta t^2}{6} V_{ttt}} + O(\Delta t^3) = \frac{2}{1+2r}\left(\cancel{V_{xx}} + \frac{1}{12}\Delta x^2 V_{(4)} + O(\Delta x^4)\right)$$
$$+ \frac{1-2r}{1+2r}\left(\cancel{-V_t} + \frac{\Delta t}{2} V_{tt} - \boxed{\frac{\Delta t^2}{6} V_{ttt}} + O(\Delta t^3)\right) + \tau^n$$

So

$$\tau^n = \underbrace{\left(1 - \frac{1-2r}{1+2r}\right)}_{\frac{2}{1+2r}} \frac{\Delta t}{2} V_{tt} + \left(1 + \frac{1-2r}{1+2r}\right)\frac{\Delta t^2}{6} V_{ttt} - \frac{2}{1+2r}\cdot\frac{1}{12}\Delta x^2 V_{(4)} + O(\Delta x^4) + O(\Delta t^3)$$

Since $V_{tt} = (V_t)_t = (V_{xx})_t = (V_t)_{xx} = V_{xxxx}$

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Then

$$\tau^n = \frac{2}{1+2r}\left(\frac{\Delta t}{2} - \frac{1}{12}\Delta x^2\right) V_{tt} + O(\Delta t^2) + O(\Delta x^4)$$

$$= \Delta x^2 \underbrace{\left(\frac{2}{1+2r}\right)\left(\frac{r}{2} - \frac{1}{12}\right)}_{\text{this is bounded for } r\in(0,\infty) \text{ so this is an } O(\Delta x^2) \text{ term}} V_{tt} + O(\Delta t^2) + O(\Delta x^4)$$

$$= O(\Delta x^2) + O(\Delta t^2).$$

HW 2.31 (e)

$$V_k^{n+1} = V_k^n - \frac{a\Delta t}{\Delta x}\left(V_{k+1}^n - V_k^n\right) + \Delta t\,\tau^n \qquad V_t + aV_x = 0$$

Expanding gives

$$V_{k+1}^n - V_k^n = \cancel{V} + \Delta x V_x + \frac{\Delta x^2}{2} V_{xx} + O(\Delta x^3) - \cancel{V}$$

Thus

$$\cancel{V} + \cancel{\Delta t} V_t + \frac{\Delta t^{\cancel{2}}}{2} V_{tt} + O(\Delta t^{\cancel{3}\,2}) = \cancel{V} - \frac{a\cancel{\Delta t}}{\cancel{\Delta x}}\left(\Delta x V_x + \frac{\Delta x^2}{2} V_{xx} + O(\Delta x^{\cancel{3}\,2})\right) + \cancel{\Delta t}\,\tau^n$$

Therefore

$$\cancel{\boxed{V_t}} + \frac{\Delta t}{2} V_{tt} + O(\Delta t^2) = -a\left(\cancel{\boxed{V_x}} + \frac{\Delta x}{2} V_{xx} + O(\Delta x^2)\right) + \tau^n$$

So

$$\tau^n = \frac{\Delta t}{2} V_{tt} + a\frac{\Delta x}{2} V_{xx} + O(\Delta x^2) + O(\Delta t^2)$$

Since $V_{tt} = -aV_{xt} = -aV_{tx} = a^2 V_{xx}$ then

$$\tau^n = \left(a^2\frac{\Delta t}{2} + a\frac{\Delta x}{2}\right)V_{xx} + O(\Delta x^2) + O(\Delta t^2)$$

$$= O(\Delta t) + O(\Delta x).$$

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HW 2.3.2 (a)

$$V_k^{n+1} = V_k^n + r\left(-\frac{1}{12}\left(V_{k-2}^n + V_{k+2}^n\right) + \frac{4}{3}\left(V_{k-1}^n + V_{k+1}^n\right) - \frac{5}{2}V_k^n\right), \quad V_t = \nu V_{xx}, \quad r = \nu\frac{\Delta t}{\Delta x^2}$$

Expand:

$$V_{k+2}^n + V_{k-2}^n = V + \cancel{2\Delta x V_x} + 2\Delta x^2 V_{xx} + \cancel{\frac{4\Delta x^3}{3} V_{(3)}} + \frac{2\Delta x^4}{3} V_{(4)} + \cancel{\frac{4\Delta x^5}{15} V_{(5)}} + O(\Delta x^6)$$
$$+ V - \cancel{''} + '' - \cancel{''} + '' - \cancel{''} + ''$$
$$= 2V + 4\Delta x^2 V_{xx} + \frac{4\Delta x^4}{3} V_{(4)} + O(\Delta x^6)$$

$$V_{k+1}^n + V_{k-1}^n = V + \Delta x V_x + \frac{\Delta x^2}{2} V_{xx} + \frac{\Delta x^3}{3!} V_{(3)} + \frac{\Delta x^4}{4!} V_{(4)} + \frac{\Delta x^5}{5!} V_{(6)} + O(\Delta x^6)$$
$$+ '' - '' + '' - '' + '' - '' + ''$$
$$= 2V + \Delta x^2 V_{xx} + \frac{1}{12}\Delta x^4 V_{(4)} + O(\Delta x^6)$$

$$-\frac{1}{12}\left('' \right) + \frac{4}{3}\left('' \right) - \frac{5}{2}V_k^n = -\frac{1}{6}V - \frac{1}{3}\Delta x^2 V_{xx} - \cancel{\frac{1}{9}\Delta x^4 V_{(4)}}$$
$$+ \frac{8}{3}V + \frac{4}{3}\Delta x^2 V_{xx} + \cancel{\frac{1}{9}\Delta x^4 V_{(4)}} - \frac{5}{2}V + O(\Delta x^6)$$
$$= \cancel{\left(-\frac{1}{6} + \frac{8}{3} - \frac{5}{2}\right)}^{0} V + \left(-\frac{1}{3} + \frac{4}{3}\right)\Delta x^2 V_{xx} + O(\Delta x^6)$$
$$= \Delta x^2 V_{xx} + O(\Delta x^6)$$

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Therefore

$$\cancel{V} + \cancel{\Delta t} V_t + \frac{\overset{1}{\cancel{\Delta t^2}}}{2} V_{tt} + O(\Delta t^{\overset{2}{\cancel{3}}}) = \cancel{V} + \overset{\frac{\nu \cancel{\Delta t}}{\Delta x^2}}{r}\left(\Delta x^2 V_{xx} + O(\Delta x^6)\right) + \Delta t\, \tau^n$$

$$\boxed{\cancel{V_t}} + \frac{\Delta t}{2} V_{tt} + O(\Delta t^2) = \boxed{\cancel{\nu V_{xx}}} + O(\Delta x^4) + \tau^n$$

So

$$\tau^n = \frac{\Delta t}{2} V_{tt} + O(\Delta t^2) + O(\Delta x^4)$$

$$= O(\Delta t^2) + O(\Delta x^4).$$

HW 2.33.(a)

$$V_k^{n+1} = V_k^{n-1} - R\delta_0 V_k^n + \Delta t\,\tau^n, \qquad V_t + aV_x = 0 \qquad R = a\frac{\Delta t}{\Delta x}$$

Expanding as a Taylor series gives

$$V_k^{n+1} = V + \Delta t V_t + \frac{\Delta t^2}{2} V_{tt} + O(\Delta t^3)$$

$$V_k^{n-1} = V - \Delta t V_t + \frac{\Delta t^2}{2} V_{tt} + O(\Delta t^3)$$

$$\delta_0 V_k^n = V_{k+1}^n - V_{k-1}^n = \cancel{V} + \Delta x V_x + \cancel{\frac{\Delta x^2}{2} V_{xx}} + \frac{\Delta x^3}{3!} V_{(3)} + \cancel{\frac{\Delta x^4}{4!} V_{(4)}} + \frac{\Delta x^5}{5!} V_{(5)} + O(\Delta x^6)$$

$$- \left(\cancel{''} - '' + \cancel{''} - '' + \cancel{''} - '' + \right)$$

$$= 2\Delta x V_x + \frac{1}{3}\Delta x^3 V_{(3)} + O(\Delta x^5)$$

Therefore

$$\cancel{V} + \Delta t V_t + \cancel{\frac{\Delta t^2}{2} V_{tt}} + O(\Delta t^3) = \cancel{V} - \Delta t V_t + \cancel{\frac{\Delta t^2}{2} V_{tt}} + O(\Delta t^3)$$

$$- R\left(2\Delta x V_x + \frac{1}{3}\Delta x^3 V_{(3)} + O(\Delta x^5)\right) + \Delta t\,\tau^n$$

$$2\cancel{\Delta t} V_t + O(\Delta t^{\cancel{3}\,2}) = -a\cancel{\Delta t}\left(2V_x + \frac{1}{3}\Delta x^2 V_{(3)} + O(\Delta x^4)\right) + \cancel{\Delta t}\,\tau^n$$

So

$$\tau^n = \boxed{\cancel{2V_t + a2V_x}} + \frac{a}{3}\Delta x^2 V_{(3)} + O(\Delta t^2) + O(\Delta x^4)$$

$$\tau^n = O(\Delta x^2) + O(\Delta t^2)$$

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HW 2.3.3.(b)

$$v_k^{n+1} = v_k^{n-1} - R\delta_0 u_k^n + \frac{R}{6}\delta^2\delta_0 u_k^n, \qquad v_t + a v_x = 0, \qquad R = \frac{a\Delta t}{\Delta x}$$

Since

$$\delta^2\delta_0 u_k^n = 2\Delta x^3 v_{(3)} + \frac{1}{2}\Delta x^5 v_{(5)} + O(\Delta x^7)$$

we obtain, following the calculation of the previous problem, that

$$\cancel{v} + \Delta t\, v_t + \cancel{\frac{\Delta t^2}{2} v_{tt}} + \frac{\Delta t^3}{3!} v_{ttt} + \cancel{\frac{\Delta t^4}{4!} v_{tttt}} + O(\Delta t^5) = \cancel{v} - \Delta t\, v_t + \cancel{\frac{\Delta t^2}{2} v_{tt}} - \frac{\Delta t^3}{3!} + \cancel{\frac{\Delta t^4}{4!}} + O(\Delta t^5)$$
$$- R\left(2\Delta x\, v_x + \frac{1}{3}\Delta x^3 v_{(3)} + \frac{2}{5!}\Delta x^5 v_{(5)} + O(\Delta x^7)\right)$$
$$+ \frac{R}{6}\left(2\Delta x^3 v_{(3)} + \frac{1}{2}\Delta x^5 v_{(5)} + O(\Delta x^7)\right) + \Delta t\,\tau^n$$

So

$$\boxed{2v_t} + \frac{1}{3}\Delta t^2 v_{ttt} + O(\Delta t^4) = -\boxed{a\left(2v_x\right.} + \frac{1}{3}\Delta x^2 v_{(3)} + \frac{1}{60}\Delta x^4 v_{(5)}\Big)$$
$$+ \frac{a}{6}\left(2\Delta x^2 v_{(3)} + \frac{1}{2}\Delta x^4 v_{(5)}\right) + O(\Delta x^6) + \tau^n$$

Therefore

$$\tau^n = \frac{1}{3}\Delta t^2 v_{ttt} + a\left(\frac{1}{60} - \frac{1}{12}\right)\Delta x^4 v_{(5)} + O(\Delta t^4) + O(\Delta x^6) = O(\Delta x^4) + O(\Delta t^2)$$

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