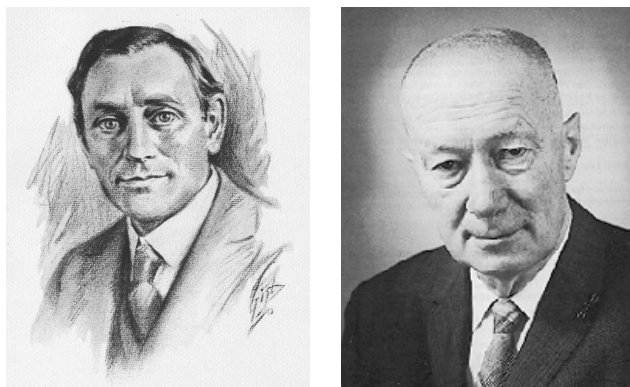


### The Viscous Burgers Equation

The viscous Burgers equation, see for example Chapter 14.3 in Iserles [1], is obtained by coupling a diffusion term such as found in the heat equation with a conservation law representing transport of a one-dimensional velocity profile by its own velocity. The resulting equation may be written

$$u_t + uu_x - \varepsilon u_{xx} = 0 \quad (1)$$

where  $\varepsilon > 0$ . When  $\varepsilon = 0$  the diffusion term drops out and the resulting conservation law is simply known as Burgers equation. These equations were introduced by Bateman in 1915 and further studied by Burgers in 1948.



Harry Bateman left; Johannes Martinus Burgers right.

For simplicity we will consider the initial value problem on the real line with  $L$ -periodic initial condition  $u_0$  such that

$$u_0(x) = u_0(x + L) \quad \text{for all } x \in \mathbf{R}.$$

Since the initial  $L$ -periodicity is preserved by the dynamics of the viscous Burgers equation (1), then the resulting solution  $u(t, x)$  further satisfies

$$u(t, x) = u(t, x + L) \quad \text{for all } t \geq 0 \text{ and } x \in \mathbf{R}.$$

### An Energy Analysis of the Viscous Burgers Equation

In this section we will assume the viscous Burgers equation is well-posed. Thus, we suppose the solution exists, is differentiable and depends continuously on the initial condition  $u_0$ . Note that the mathematical proof of this well-posedness relies critically on the fact that  $\varepsilon > 0$ . In particular, when  $\varepsilon = 0$  the resulting solution  $u(t, x)$  is known to develop shocks or jump discontinuities in finite time, even if the initial condition is infinitely smooth.

We begin our analysis by multiplying equation (1) through by  $u$  and integrating over the fundamental domain  $[0, L]$  in space. Interchanging the time derivative with the spatial integral yields

$$\int_0^L uu_t = \int_0^L \frac{1}{2} \frac{\partial u^2}{\partial t} = \frac{1}{2} \frac{d}{dt} \int_0^L u^2$$

where the interchange of the respective limiting processes may be justified by the already assumed differentiability of the solution  $u(t, x)$ . We remark that the partial derivatives in time may be written as regular derivatives when they are moved outside because the value of the integrated function does not depend on  $x$ .

Integrating by parts and using the  $L$ -periodicity of  $u$  implies

$$\int_0^L u^2 u_x = u^3 \Big|_0^L - \int_0^L 2u^2 u_x = - \int_0^L 2u^2 u_x.$$

Therefore,

$$\int_0^L u^2 u_x = 0.$$

The fact the nonlinear term cancels out in the above integral is not surprising but merely a confirmation that this part of the equation came from a conservation law.

A similar treatment of the viscous term yields

$$-\varepsilon \int_0^L uu_x x = -\varepsilon uu_x \Big|_0^L + \varepsilon \int_0^L u_x^2 = \varepsilon \int_0^L u_x^2.$$

This term will later be estimated by what is known as the Poincaré inequality.

Before that we summarize what we have obtained by combining the above calculations to obtain the energy equality

$$\frac{1}{2} \frac{d}{dt} \int_0^L u^2 + \varepsilon \int_0^L u_x^2 = 0. \quad (2)$$

The reason this is called an energy equality is two-fold. First, upon remembering that  $u$  represents a velocity and that kinetic energy is one-half mass times velocity squared, the integral

$$E_K = \frac{1}{2} \int_0^L u^2$$

then represents a sum proportional to the physical kinetic energy. On the other hand, mathematically setting  $\varepsilon = 0$  results in  $dE_K/dt = 0$ , which shows in the absence of viscous dissipation that  $E_K$  is constant.

We are now ready to continue our analysis by using

**Poincaré's Inequality.** *Let  $f: \mathbf{R} \rightarrow \mathbf{R}$  be a continuously differentiable function. It holds that*

$$\int_0^L |f(x) - V|^2 dx \leq L^2 \int_0^L |f'(x)|^2 dx \quad \text{where} \quad V = \frac{1}{L} \int_0^L f(x) dx.$$

*Proof.* By the intermediate value theorem there is a point  $s \in [0, L]$  such that  $f(s) = V$ . Now, by the Fundamental Theorem of Calculus followed by the Cauchy–Schwarz inequality we have for  $x \in [0, L]$  that

$$\begin{aligned} |f(x) - V| &= |f(x) - f(s)| = \left| \int_s^x f'(\xi) d\xi \right| \\ &\leq \left| \int_s^x d\xi \right|^{1/2} \left| \int_s^x |f'(\xi)|^2 d\xi \right|^{1/2} \leq L^{1/2} \left| \int_0^L |f'(\xi)|^2 d\xi \right|^{1/2}. \end{aligned}$$

Squaring both sides and integrating in  $x$  now obtains

$$\int_0^L |f(x) - V|^2 dx \leq L \int_0^L \int_0^L |f'(\xi)|^2 d\xi dx = L^2 \int_0^L |f'(x)|^2 dx.$$

This finishes the proof of the inequality. ////

To finish out energy analysis of the viscous Burgers equation define

$$Y = \int_0^L |u - V|^2 \quad \text{where} \quad V = \frac{1}{L} \int_0^L u$$

Since  $u$  is  $L$ -periodic, so are its derivatives. Therefore, integrating the viscous Burgers equation (1) yields

$$\frac{d}{dt} \int_0^L u = \int_0^L u_t = \int_0^L (-uu_x + \varepsilon u_{xx}) = \left( -\frac{1}{2}u^2 + \varepsilon u_x \right) \Big|_0^L = 0$$

Consequently, the average value  $V$  of  $u$  doesn't depend on time and is therefore constant. We obtain

$$\begin{aligned} \frac{dY}{dt} &= \frac{d}{dt} \int_0^L |u - V|^2 = \frac{d}{dt} \int_0^L (u^2 - 2Vu + V^2) \\ &= \frac{d}{dt} \left( \int_0^L u^2 + LV^2 \right) = \frac{d}{dt} \int_0^L u^2. \end{aligned}$$

Taking  $f(x) = u(t, x)$  in Poincaré's inequality and substituting into (2) results in the differential inequality

$$\frac{1}{2} \frac{dY}{dt} + \frac{\varepsilon}{L^2} Y \leq 0. \tag{3}$$

Multiply by  $\mu = \exp(2\varepsilon L^{-2}t)$  and integrate over  $[0, T]$  to obtain  $Y(T) = Y(0)e^{-2\varepsilon L^{-2}t}$  or equivalently that

$$\int_0^L |u - V|^2 \leq M e^{-\alpha t} \quad \text{where} \quad M = \int_0^L |u_0 - V|^2 \quad \text{and} \quad \alpha = 2\varepsilon/L^2. \tag{4}$$

Note that the above way of converting the differential inequality (3) into the inequality (4) is often attributed to Grönwall and called Grönwall's inequality.

The above energy analysis allows us to conclude that  $u(t, x)$  converges exponentially over time to a constant function in  $x$ . That constant is furthermore exactly equal to the average velocity  $V$  of the initial condition  $u_0$ .

### Fourier Discretization of the Viscous Burgers Equation

Since  $u(t, x)$  is  $L$ -periodic in space at any fixed point in time, it shall be convenient to rescale the  $x$ -axis so that  $L = 2\pi$  and represent  $u(t, x)$  in terms of the Fourier series

$$u(t, x) = \sum_{k=-\infty}^{\infty} a_k(t) e^{-ikx}.$$

We further truncate this series to obtain the discretization

$$u(t, x) \approx \sum_{k=-K/2}^{K/2} a_k(t) e^{-ikx}.$$

As it will be numerically efficient to work with discrete transforms of size  $N = 2^n$  we further suppose  $K < N$  and take

$$a_k(t) = 0 \quad \text{when} \quad k > K/2 \quad \text{or} \quad k < -K/2$$

to obtain that

$$\sum_{k=-K/2}^{K/2} a_k(t) e^{-ikx} = \sum_{k=-N/2}^{N/2-1} a_k(t) e^{-ikx}.$$

Upon taking  $x_\ell = 2\pi\ell/N$  the discrete Fourier inversion theorem now implies that

$$a_k(0) = \frac{1}{N} \sum_{\ell=0}^{N-1} u_0(x_\ell) e^{ikx_\ell} \quad \text{for} \quad k = -K/2, \dots, K/2$$

where again  $a_k(0) = 0$  for  $k = -N/2, \dots, -K/2 - 1$  and  $k = K/2 + 1, \dots, N/2 - 1$ .

We now approximate each of the terms involving  $u(t, x)$  from the viscous Burgers equation (1) in terms of the Fourier coefficients  $a(t)$ .

$$\begin{aligned} u_t &\approx \frac{\partial}{\partial t} \sum_{k=-K/2}^{K/2} a_k(t) e^{-ikx} = \sum_{k=-K/2}^{K/2} a'_k(t) e^{-ikx}. \\ uu_x &\approx \sum_{j=-K/2}^{K/2} a_j(t) e^{-ijx} \frac{\partial}{\partial x} \sum_{k=-K/2}^{K/2} a_k(t) e^{-ikx} \\ &= \sum_{k=-K/2}^{K/2} \sum_{j=-K/2}^{K/2} a_j(t) (-ik) a_k(t) e^{-i(k+j)x} \\ &= \sum_{k=-K/2}^{K/2} \sum_{p=k-K/2}^{k+K/2} (-ik) a_k(t) a_{p-k}(t) e^{-ipx} = \sum_{p=-K}^K c_p e^{-ipx} \end{aligned}$$

where we have made the change of variables  $p = k + j$  and

$$c_p = \sum_{k=\max(-K/2, p-K/2)}^{\min(K/2, p+K/2)} (-ik) a_k(t) a_{p-k}(t) \quad \text{for} \quad p = -K, \dots, K. \quad (5)$$

Finally,

$$u_{xx} = \frac{\partial^2}{\partial x^2} \sum_{k=-K/2}^{K/2} a_k(t) e^{-ikx} = - \sum_{k=-K/2}^{K/2} k^2 a_k(t) e^{-ikx}.$$

Observe that directly computing all the  $c_p$  coefficients using (5) takes  $\mathcal{O}(K^2)$  number of operations. However, by making use of the fast Fourier transform, this can be reduced to  $\mathcal{O}(N \log N)$ . Since

$$uu_x = \frac{1}{2} \frac{\partial u^2}{\partial x}$$

first transform the coefficients  $a_k(t)$  to obtain the values of  $u(x_\ell)$  for  $\ell = 0, \dots, N-1$  in  $\mathcal{O}(N \log N)$  operations. Forming  $u(x_\ell)^2 = u^2(x_\ell)$  and dividing by 2 takes  $N$  operations. Finally, in another  $\mathcal{O}(N \log N)$  operations the Fourier inversion theorem yields that

$$u^2 \approx \sum_{k=-N/2}^{N/2-1} b_k e^{-ikx} \quad \text{where} \quad b_k = \frac{1}{N} \sum_{\ell=0}^{N-1} u^2(x_\ell) e^{ikx_\ell}. \quad (6)$$

It follows that  $c_p = -ipb_p$  modulo a couple conditions which we now discuss.

Observe first that the coefficients  $b_p$  given by (5) range for  $p = -K, \dots, K$  which is twice the number of non-zero Fourier modes present in the original approximation of  $u$ . Therefore, to compute these coefficients accurately using (6) we must have that  $N \geq 2K$ . This is the condition derived in class to avoid aliasing in the Fourier representation of  $u^2$ .

Since we will only use the values of  $b_k$  for  $k = -K/2, \dots, K/2$  when integrating  $u$ , a slight modification of the aliasing argument leads to a more efficient use of computer resources. Upon examining the terms which appear in (6) it is clear that the aliasing which occurs as  $N$  becomes less than  $2K$  is only in the high modes. Specifically, the modes of the form  $e^{-ipx_\ell}$  are alias according to the rules.

$$\begin{aligned} e^{-ipx_\ell} & \text{ is aliased to } e^{-i(p-N)x_\ell} & \text{ for } p \geq N/2 \\ e^{-ipx_\ell} & \text{ is aliased to } e^{-i(p+N)x_\ell} & \text{ for } p < -N/2. \end{aligned}$$

Now, suppose that  $K = \alpha N$  for some  $\alpha > 1/2$ . Since  $|k| \leq K/2$  and  $|j| \leq K/2$  it follows that  $|p| \leq \alpha N$ . Note that

$$\begin{aligned} N/2 \leq p \leq \alpha N & \text{ implies } -N/2 < p - N \leq -(1 - \alpha)N \\ -\alpha N \leq p < -N/2 & \text{ implies } (1 - \alpha)N < p + N \leq N/2 \end{aligned}$$

Therefore only modes such that  $|p| \geq \min(\alpha N, (1 - \alpha)N)$  are aliased. Since we only need the values of  $b_k$  for  $|p| \leq K/2$ . We choose  $\alpha > 0$  to be the smallest value such that

$$\min(\alpha N, (1 - \alpha)N) < K/2 = \alpha N/2.$$

Therefore

$$1 - \alpha < \alpha/2 \quad \text{which implies} \quad \alpha > 2/3.$$

This is sometimes called the 2/3 anti-aliasing rule for a quadratic nonlinearity.

### The Reality Condition for the Fourier Series

Since the solution  $u$  to the viscous Burgers equation represents a velocity field then it must be real valued. This implies a certain symmetry condition on the coefficients  $a_k$  of the Fourier series. Suppose the approximation

$$u(x, t) \approx \sum_{k=-K/2}^{K/2} a_k(t) e^{-ikx}$$

is real valued. Taking complex conjugates yields

$$\overline{\sum_{k=-K/2}^{K/2} a_k(t) e^{-ikx}} = \sum_{k=-K/2}^{K/2} \overline{a_k(t)} e^{ikx} = \sum_{p=-K/2}^{K/2} \overline{a_{-p}(t)} e^{-ipx}.$$

after the change of variables  $p = -k$ . Now, since the Fourier modes are linearly independent, it follows that

$$a_k(t) = \overline{a_{-k}(t)} \quad \text{for} \quad k = -K/2, \dots, K/2 \quad \text{and} \quad t \geq 0. \quad (7)$$

The good news is that this symmetry condition allows for the creation of a specialized real-valued Fourier transform that economizes on the number of floating-point operations needed by a factor of two. The bad news is that rounding errors in the standard complex transform can cause the symmetric condition to be violated. For this reason it is important when taking the Fourier transform of coefficients which satisfy (7) to always set the imaginary part of the resulting velocity field  $u$  to zero. As the energy analysis performed earlier only works for real-valued velocity fields, then rounding errors in the imaginary part of  $u$  may grow catastrophically without bound if not explicitly set to zero.

### MATLAB Code

In this section we present a simple script written in MATLAB which approximates solutions to the viscous Burgers equation with  $2\pi$ -periodic boundary conditions using the Runge–Kutta second order method. By taking Fourier transforms of the partial differential equation we have obtained an ordinary differential equation of the form

$$\frac{dy}{dt} = F(y) \quad \text{where} \quad y = (a_{-K/2}, \dots, a_{K/2}) \quad (8)$$

and

$$F(y) = -(i(-K/2)b_{-K/2} + \varepsilon(-K/2)^2 a_{-K/2}, \dots, i(K/2)b_{K/2} + \varepsilon(K/2)^2 a_{K/2})$$

with the  $b_k$  given by equation (6).

Setting  $t_j = jh$  where  $h > 0$  is the size of the time steps, a second order method can be obtained by integrating both sides of (8) over the interval  $[t_j, t_{j+1}]$  and applying the trapezoid rule to approximate the resulting integral. Thus,

$$y(t_{j+1}) - y(t_j) = \int_{t_j}^{t_{j+1}} F(y(t)) dt \approx \frac{h}{2} (F(y(t_j)) + F(y(t_{j+1}))).$$

To obtain an explicit scheme we further make a first order approximation of  $y(t_{j+1})$  appearing in the term  $F(y(t_{j+1}))$  on the right-hand side via an Euler step of the form

$$y(t_{j+1}) \approx y(t_j) + hF(y(t_j)).$$

Now, denoting by  $y_j$  the approximation such that  $y_j \approx y(t_{j+1})$  we obtain the rule

$$\begin{aligned} k_1 &= hF(y_j) \\ k_2 &= hF(y_j + k_1) \\ y_{j+1} &= y_j + (k_1 + k_2)/2 \end{aligned}$$

which is often referred to as RK2.

A MATLAB script to implement this method is

```

1 clear all
2 global kvec filt epsilon
3 N=256;
4 K=2*floor(N/3);
5 kvec=[0:K/2,zeros(1,N-K-1),-K/2:-1];
6 filt=[ones(1,K/2+1),zeros(1,N-K-1),ones(1,K/2)];
7 epsilon=0.01;
8 Tfin=1;
9
10 x=[0:2*pi/N:2*pi-2*pi/N];
11 load('initial.dat');
12 y=initial(:,1)';
13 a0=ifft(y).*filt;
14 yold=y;
15 steps=4096;
16
17 h=Tfin/steps;
18 at=a0;
19 n=0;
20 for j=1:steps
21     k1=h*f(at);
22     k2=h*f(at+k1);
23     at=at+(k1+k2)/2;
24     n=n+1;
25 end
26 yt=real(fft(at));
27 plot(x,yt);

```

where the file `f.m` is given by

```

1 function ft = f(at)
2     global kvec filt epsilon

```

```
3     u=real(fft(at.*filt));
4     u2=u.*u/2;
5     b=ifft(u2);
6     ft=-kvec.*kvec*epsilon.*at+1i*kvec.*b;
7 end
```

In the above code the file `initial.dat` should consist of a list of  $N = 256$  numbers that correspond to the values of initial condition  $u_0(x_\ell)$  on the fundamental domain  $[0, 2\pi]$ .

### References

1. Arieh Iserles, *A First Course in the Numerical Analysis of Differential Equations*, Cambridge Texts in Applied Mathematics, 1996.



Your answers should be presented in the form of a written report with source code, graphs, tables and program output where appropriate. Style of presentation counts as well as spelling, punctuation and grammar. Please work independently; however, it is fine to visit the UNR Writing Center for help with writing style. If you have any difficulties please talk with me in my office hours or set up an appointment.

1. Plot the points in the data file that correspond to your initial condition.
2. Energy analysis shows that the velocity profile given by the solution  $u(t, x)$  to the viscous Burgers equation converges to a constant as  $t \rightarrow \infty$ . Find that constant for your initial condition.
3. For Fourier transforms of length  $N = 256$  show that the 2/3 anti-aliasing rule gives a cutoff with  $K/2 = 85$ .
4. Use the RK2 method with a time step of  $h = 1/128$  and  $\varepsilon = 0.01$  to compute an approximation of  $u(t, x)$  at time  $t = 1$ . You may use the code developed in class, the code included with this handout or write your own. Plot your approximation.
5. Given a time step of size  $h = 2^{-m}$  let  $U_m(x_\ell)$  be the corresponding approximation of  $u(t, x)$  at time  $t = 1$ . Compute the norms of the errors

$$E_m = \|U_m - U_{m+1}\| \quad \text{for} \quad m = 6, \dots, 11.$$

and make a table showing  $E_m$  versus  $m$ .

6. Form the ratios  $E_m/E_{m+1}$  from the values found in the previous problem to verify that your implementation of the RK2 method is actually second order. Explain the reasoning behind your verification.
7. [Extra Credit] Modify your code to use the RK4 method and repeat questions 5 and 6 above to verify that your implementation is fourth order.
8. [Extra Credit] Derive an anti-aliasing rule similar to the 2/3 rule that works for cubic nonlinearities such as  $u^3$ .