1. Let $I_n = (-1/n, 1/n)$ for $n \in \mathbf{N}$ and $U = \bigcap_{n=1}^{\infty} I_n$. Then

(A) $U = [-1, 1]$

(B) $U = (-1, 1)$

(C) $U = \{ 0 \}$

(D) $U = \emptyset$

(E) none of these

2. Let $f\colon \mathbf{R} \to \mathbf{R}$. If $A \subseteq \mathbf{R}$ then

(A) $\big(f^{-1}(A)\big)^c = \{ f(x) : x \notin A \}$

(B) $\big(f^{-1}(A)\big)^c = \{ f(x) : x \in A \}$

(C) $\big(f^{-1}(A)\big)^c = \{ x \in \mathbf{R} : f(x) \notin A \}$

(D) $\big(f^{-1}(A)\big)^c = \{ x \in \mathbf{R} : f(x) \in A \}$

(E) none of these

3. The set $\mathcal{S}$ of simple functions is equal to

(A) $\mathcal{S} = \Big\{ \sum_{k=1}^{n} a_k \chi_{E_k} : \quad n \in \mathbf{N}, \quad a_k \in \mathbf{R} \quad \text{and} \quad E_k \in \mathcal{M} \Big\}$

(B) $\mathcal{S} = \Big\{ \sum_{k=1}^{n} a_k \chi_{E_k} : \quad n \in \mathbf{N}, \quad a_k \geq 0 \quad \text{and} \quad E_k \in \mathcal{M} \Big\}$

(C) $\mathcal{S} = \Big\{ \sum_{k=1}^{\infty} a_k \chi_{E_k} : \quad a_k \in \mathbf{R} \quad \text{and} \quad E_k \in \mathcal{M} \Big\}$

(D) $\mathcal{S} = \Big\{ \sum_{k=1}^{\infty} a_k \chi_{E_k} : \quad a_k \geq 0 \quad \text{and} \quad E_k \in \mathcal{M} \Big\}$

(E) none of these

4. A set $F \subseteq \mathbf{R}$ is closed if $F = \overline{F}$. This is equivalent to saying F is closed if

(A) for every $x \in F$ there exists $r > 0$ such that $[x - r, x + r] \subseteq F$

(B) for every sequence $x_n \in F$ and $x \in \mathbf{R}$ then $x_n \to x$ implies $x \in F$

(C) for every sequence $x_n \in F^c$ and $x \in \mathbf{R}$ then $x_n \to x$ implies $x \in F^c$

(D) every sequence $x_n \in F$ has a convergent subsequence

(E) none of these

5. Let $D \subseteq \mathbf{R}$ and $f\colon D \to \mathbf{R}$. Suppose for every $a \in D$ and $\epsilon > 0$ there is $\delta > 0$ such that $b \in D$ and $|a - b| < \delta$ implies $|f(a) - f(b)| < \epsilon$. Then f must be

(A) continuous

(B) uniformly continuous

(C) differentiable

(D) both (A) and (B)

(E) both (A), (B) and (C)

6. Let $D \subseteq \mathbf{R}$ and $f_n\colon D \to \mathbf{R}$ for $n \in \mathbf{N}$. Suppose for every $\epsilon > 0$ there is $N \in \mathbf{N}$ such that $n, m \geq N$ and $x \in D$ implies $|f_n(x) - f_m(x)| < \epsilon$. Then the sequence f_n of real valued functions must be

(A) pointwise convergent

(B) uniformly convergent

(C) differentiable

(D) both (A) and (B)

(E) both (A), (B) and (C)

7. Given an interval I let $\ell(I)$ be its length. For each subset $A \subseteq \mathbf{R}$, the Lebesgue outer measure of A, denoted by $\lambda^*(A)$, is defined by

(A) $\lambda^*(A) = \inf\left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals where } A \subseteq \bigcap_n I_n \right\}$

(B) $\lambda^*(A) = \inf\left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals where } A \subseteq \bigcup_n I_n \right\}$

(C) $\lambda^*(A) = \sup\left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals where } A \supseteq \bigcap_n I_n \right\}$

(D) $\lambda^*(A) = \sup\left\{ \sum_n \ell(I_n) : I_n \text{ are open intervals where } A \supseteq \bigcup_n I_n \right\}$

(E) none of these

8. Let $\hat{C}$ be the collection of Borel measurable functions, $\mathcal{B}$ the set of Borel measurable sets and τ the collection of open sets in $\mathbf{R}$. Then

(A) $\hat{C} = \{\, f\colon \mathbf{R} \to \mathbf{R} \text{ such that } f^{-1}(B) \in \mathcal{B} \text{ for every } B \in \mathcal{B} \,\}$

(B) $\hat{C} = \{\, f\colon \mathbf{R} \to \mathbf{R} \text{ such that } f^{-1}(O) \in \mathcal{B} \text{ for every } O \in \tau \,\}$

(C) $\hat{C} = \{\, f\colon \mathbf{R} \to \mathbf{R} \text{ such that } f^{-1}(a) \in \mathcal{B} \text{ for every } a \in \mathbf{R} \,\}$

(D) both (A) and (B)

(E) both (A), (B) and (C)

9. The collection of Lebesgue mesaureable sets $\mathcal{M}$ is defined to be

(A) $\mathcal{M} = \{\, E \subseteq \mathbf{R} : \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \cap E^c) \text{ for every } W \subseteq \mathbf{R}\,\}$

(B) $\mathcal{M} = \{\, E \subseteq \mathbf{R} : \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W^c \cap E) \text{ for every } E \subseteq \mathbf{R}\,\}$

(C) $\mathcal{M} = \{\, W \subseteq \mathbf{R} : \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W \cap E^c) \text{ for every } E \subseteq \mathbf{R}\,\}$

(D) $\mathcal{M} = \{\, W \subseteq \mathbf{R} : \lambda^*(W) = \lambda^*(W \cap E) + \lambda^*(W^c \cap E) \text{ for every } E \subseteq \mathbf{R}\,\}$

(E) none of these

10. In the following true false questions $\hat{C}$ is the set of Borel measurable functions, $\mathcal{B}$ is the collection of Borel measurable sets, λ^* is the Lebesgue outer measure and $\mathcal{M}$ is the set of Lebesgue measurable sets.

(i) If $f\colon \mathbf{R} \to \mathbf{R}$ is Lebesgue measureable then is must be Borel measurable.

(A) true

(B) false

(ii) Suppose

$$x = \sum_{k=1}^{\infty} \frac{d_k}{10^k} \quad \text{and} \quad y = \sum_{k=1}^{\infty} \frac{e_k}{10^k} \quad \text{where} \quad d_k, e_k \in \{0, 1, 2, \ldots, 9\}.$$

If $x \le y$ then $d_1 \le e_1$.

(A) true

(B) false

(iii) If $\lambda^*(E) = 0$ then $E \in \mathcal{M}$.

(A) true

(B) false

11. Let $\mathcal{S}$ be the set of simple functions. The Lebesgue integral of a positive function $f\colon \mathbf{R} \to \mathbf{R}$ is defined by

(A) $\displaystyle\int f = \sup\Big\{ \int s : \quad s \in \mathcal{S} \quad \text{and} \quad 0 \le s \le f \Big\}$

(B) $\displaystyle\int f = \inf\Big\{ \int s : \quad s \in \mathcal{S} \quad \text{and} \quad f \le s \Big\}$

(C) both (A) and (B)

(D) none of these

12. Show there is an irrational number between any two rational numbers.

13. Suppose $f\colon [0,1] \to \mathbf{R}$ is continuous and $f(c) > 0$ for some $c \in (0,1)$. Show there is $h > 0$ such that $|x - c| < h$ implies $f(x) > 0$.

14. State and prove Fatou's Lemma.

15. State and prove the Dominated Convergence Theorem.

16. Let $d(A,B) = \inf\{\, |a-b| : a \in A \text{ and } b \in B \,\}$ and $\mathcal{M}$ be the collection of Lebesgue measurable sets. Prove one of the following:

(i) $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$ for $A, B \subseteq \mathbf{R}$ with $d(A,B) > 0$.

(ii) $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$ for $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$.

17. Let $E \subseteq \mathbf{R}$ be a closed set and define

$$F = \left\{ x \in E : \text{there exists } \epsilon > 0 \text{ such that } (x - \epsilon, x) \cap E = \emptyset \right\}.$$

Prove or disprove the claim that F is countable.

18. Prove or disprove one of the following:

(i) For $f, g \in \hat{C}$ define $f \equiv g$ to mean that $\lambda\big(\{\, x \in \mathbf{R} : f(x) \neq g(x) \,\}\big) = 0$. Prove or disprove the claim that $\equiv$ is an equivalence relation.

(ii) For $A, B \in \mathcal{M}$ define $A \simeq B$ to mean that $\lambda\big((A \setminus B) \cup (B \setminus A)\big) = 0$. Prove or disprove the claim that $\simeq$ is an equivalence relation.

19. For each $n \in \mathbf{N}$ define $E_n = [-2^n, 2^n]$ and let $f\colon \mathbf{R} \to \mathbf{R}$ be given by

$$f = \sum_{n=1}^{\infty} e^{-n} \chi_{E_n}.$$

Show that f is a well-defined non-negative Lebesgue measurable function and use the Monotone Convergence Theorem to find $\int f$.