Math 713 HW#1

#1 Let $f:\Omega \to \Lambda$ and $B \subseteq \Lambda$. Then $f^{-1}(B^c) = f^{-1}(B)^c$.

Since every step in showing $\subseteq$ and $\supseteq$ is a biconditional it is easiest to prove this as

$$\begin{aligned} f^{-1}(B^c) &= \{x \in \Omega : f(x) \in B^c\} \\ &= \{x \in \Omega : f(x) \notin B\} \\ &= \{x \in \Omega : f(x) \in B\}^c = f^{-1}(B)^c \end{aligned}$$

It is also correct to write a longer element by element proof as follows:

Claim: $x \notin f^{-1}(B)$ if and only if $f(x) \notin B$.

Proof of claim: By definition

$$f^{-1}(B) = \{x \in \Omega : f(x) \in B\}.$$

Therefore $x \in f^{-1}(B)$ if and only if $f(x) \in B$. Taking the contrapositive of this statement proves the claim.

To finish we show first that $f^{-1}(B^c) \subseteq f^{-1}(B)^c$ and then that $f^{-1}(B)^c \subseteq f^{-1}(B^c)$.

"$\subseteq$" Let $x \in f^{-1}(B^c)$. Then $f(x) \in B^c$ so $f(x) \notin B$. By the claim $x \notin f^{-1}(B)$ and so $x \in f^{-1}(B)^c$.

"$\supseteq$" Let $x \in f^{-1}(B)^c$. Then $x \notin f^{-1}(B)$. By the claim $f(x) \notin B$. Therefore $f(x) \in B^c$ and so $x \in f^{-1}(B^c)$.

#2 Let $f: X \to Y$ and $A \subseteq X$. Prove or find a counter-example to the claim $f(A^c) = (f(A))^c$.

The claim is not correct. Here is a counterexample:

$f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1$ and $A = \{1\}$.

Then

$$f(A^c) = f(\{1\}^c) = f((-\infty, 1) \cup (1, \infty)) = \{1\}.$$

whereas

$$f(A)^c = f(\{1\})^c = \{1\}^c = (-\infty, 1) \cup (1, \infty).$$

One might think that if f is one-to-one the result is true, but consider

$f: \mathbb{R} \to \mathbb{R}$ defined $f(x) = e^x$ and $A = (-\infty, 0)$.

Then

$$f(A^c) = f([0, \infty)) = [1, \infty)$$

whereas

$$f(A)^c = f((-\infty, 0))^c = (0, 1)^c = (-\infty, 0] \cup [1, \infty)$$

Indeed $f: X \to Y$ in general should be bijective in order that $f(A^c) = (f(A))^c$ hold true.

\# 3. Let $f: A \to B$ and $g: B \to A$ where $A = [0,1]$ and $B = [0,2]$ be given by $f(x) = 1+x$ and $g(y) = y/3$. Define $\tau(E) = g(f(E)^c)^c$ for $E \subseteq A$,

$$\mathcal{C} = \{E \subseteq A : E \subseteq \tau(E)\}, \quad G = \bigcup_{E \in \mathcal{C}} E \text{ and}$$

$h: A \to B$ by

$$h(x) = \begin{cases} f(x) & \text{for } x \in G \\ g^{-1}(x) & \text{for } x \notin G \end{cases}$$

Compute $h(0)$, $h(1/4)$, $h(1/2)$, $h(3/4)$ and $h(1)$.

We need G to compute h. To find G we first look for elements of $\mathcal{C}$ of the form $[a,b]$.

Suppose $E \in \mathcal{C}$ and $E = [a,b]$. Then $[a,b] \subseteq [0,1]$ implies $0 \le a \le b \le 1$. Moreover

$$[a,b] \subseteq \tau([a,b]) = g(f([a,b])^c)^c$$

$$= g([a+1,b+1]^c)^c = g([0,a+1) \cup (b+1,2])^c$$

$$= \left([0, \tfrac{a+1}{3}) \cup (\tfrac{b+1}{3}, \tfrac{2}{3}]\right)^c = [\tfrac{a+1}{3}, \tfrac{b+1}{3}] \cup (\tfrac{2}{3}, 1].$$

$$= \begin{cases} [\frac{a+1}{3}, \frac{b+1}{3}] \cup (\frac{2}{3}, 1] & \text{if } b < 1 \\ [\frac{a+1}{3}, 1] & \text{if } b = 1 \end{cases}$$

Taking the case $b=1$ we obtain $[a,1] \subseteq [\frac{a+1}{3}, 1]$ and therefore $a \ge \frac{a+1}{3}$ or $2a \ge 1$ or $a \ge \frac{1}{2}$. It follows that $[\frac{1}{2}, 1] \in E$ and therefore $[\frac{1}{2}, 1] \subseteq G$.

\# 3 continues...

Since $[\frac{1}{2},1] \subseteq G$ we know

$$h(x) = f(x) \text{ for every } x \in [\tfrac{1}{2},1].$$

It follows that

$$h(\tfrac{1}{2}) = \tfrac{3}{2},\ h(\tfrac{3}{4}) = \tfrac{9}{4} \text{ and } h(1) = 2.$$

It remains to compute $h(0)$ and $h(1/4)$. Since

$$G = \tau(G) \subseteq \tau(A) = g(f([0,1])^c)^c$$

$$= g([1,2]^c)^c = g([0,1))^c = [0,1/3)^c = [1/3,1].$$

then $0 \notin [1/3,1]$ and $1/4 \notin [1/3,1]$ implies that

$$0 \notin G \text{ and } 1/4 \notin G.$$

Therefore

$$h(0) = g^{-1}(0) = 0$$

and

$$h(1/4) = g^{-1}(1/4) = 3/4.$$

Note that it was possible to compute h for the values of $0, 1/4, 1/2, 3/4$ and 1 only knowing that

$$[\tfrac{1}{2},1] \subseteq G \subseteq [\tfrac{1}{3},1].$$

and not exactly what G is. That $G = [\frac{1}{2},1]$ seems plausible, but needs more justification.

#4. Suppose A and B are sets and that there are onto functions $f: A \to B$ and $g: B \to A$. Prove or find a counterexample to the claim that $A \sim B$.

The claim is true. Here is the <u>proof</u>: Let

$$\mathcal{C} = \{ f^{-1}(\{y\}) : y \in B \}$$

Since f is onto then each element $f^{-1}(\{y\})$ of $\mathcal{C}$ is a non-empty set. By the axiom of choice there exists a function $w: \mathcal{C} \to A$ such that $w(f^{-1}(\{y\})) \in f^{-1}(\{y\})$ for each $y \in B$. Define $\varphi: B \to A$ as

$$\varphi(y) = w(f^{-1}(\{y\})).$$

<u>Claim</u> that φ is one-to-one.

<u>Proof of the claim</u>: Suppose $y_1, y_2 \in B$. If $y_1 \neq y_2$ then

$$f^{-1}(\{y_1\}) = \{ x \in A : f(x) = y_1 \} \text{ and}$$
$$f^{-1}(\{y_2\}) = \{ x \in A : f(x) = y_2 \}$$

are disjoint. Since $f^{-1}(\{y_1\}) \cap f^{-1}(\{y_2\}) = \emptyset$ and

$$\varphi(y_1) \in f^{-1}(\{y_1\}) \text{ and } \varphi(y_2) \in f^{-1}(\{y_2\})$$

then $\varphi(y_1) \neq \varphi(y_2)$ thus proving the claim.

Similarly there is $\psi: A \to B$ based on g that is one-to-one.

Therefore the Schröder-Bernstein theorem applied to $\varphi: B \to A$ and $\psi: A \to B$ shows that $A \sim B$.

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#5 Show if a collection $\mathcal{A}$ of subsets is closed under complemetation and intersection it is an algebra.

Proof: Since $\mathcal{A}$ is already closed under complementation it is sufficient to prove that $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$.

Let $A, B \in A$. Then $A^c, B^c \in A$ and so $A^c \cap B^c \in A$. By DeMorgans law $A^c \cap B^c = (A \cup B)^c$. In other words $(A \cup B)^c \in A$ and therefore $A \cup B \in \mathcal{A}$.

\# 6 Prove that an algebra of subsets of Ω is a σ-algebra if and only if it is a monotone class.

"$\Rightarrow$" Let $\mathcal{A}$ be an algebra of subsets of Ω which is also a σ-algebra. To show $\mathcal{A}$ is a monotone class we need show that countable monotone unions and intersections are contained in $\mathcal{A}$.

Let $A_i \in \mathcal{A}$ with $A_1 \subseteq A_2 \subseteq \cdots$. Then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ because $\mathcal{A}$ is a σ-algebra and σ-algebras contain any countable union.

Let $A_i \in \mathcal{A}$ with $A_1 \supseteq A_2 \supseteq \cdots$. Then $A_1^c \subseteq A_2^c \subseteq \cdots$ and $A_i^c \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra $\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{A}$.

By DeMorgan's Law

$$\bigcup_{i=1}^{\infty} A_i^c = \left(\bigcap_{i=1}^{\infty} A_i\right)^c$$

or, in otherwords, $\left(\bigcap_{i=1}^{\infty} A_i\right)^c \in \mathcal{A}$. Thus $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$.

"$\Leftarrow$" Let $\mathcal{A}$ be an algebra of subsets of Ω which is also a monotone class. To show $\mathcal{A}$ is a σ-algebra we need to show that any countable union is contained in $\mathcal{A}$.

Let $A_i \in \mathcal{A}$. Define $B_i = \bigcup_{k=1}^{i} A_i$. Since $\mathcal{A}$ is an algebra it is closed under finite unions. Therefore $B_i \in \mathcal{A}$. Moreover since $B_1 \subseteq B_2 \subseteq \cdots$, Then $\bigcup_{i=1}^{\infty} B_i \in \mathcal{A}$ since $\mathcal{A}$ is a monotone class. Since $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$ we have shown that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

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