Math 713 HW#2

#1. Let X be a finite set and D a collection of subsets of X. Prove or find a counter example to the claim that D must be a monotone class.

The claim is true. Here is a proof.

Let M be the number of elements in X.

Let $A_n \in D$ be such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$.

Claim that the sequence A_n is eventually constant. If it were not then for each $N \geq 0$ there is $n \geq N$ such that $A_n \neq A_{n+1}$. In particular, if the sequence were not eventually constant then there would be a subsequence n_k such that $A_{n_k} \neq A_{n_{k+1}}$ for all $k \in \mathbb{N}$.

Let N_k be the number of elements in A_{n_k}.

Claim $N_k \geq k-1$.

We prove this by induction.

Clearly the number of elements in A_{n_1} is greater that or equal zero. Thus $N_1 \geq 0$ and the induction hypothesis is satisfied for $k=1$.

Now suppose, for induction, that $N_k \geq k-1$. To complete the induction we need show that $N_{k+1} \geq k$.

Since $A_{n_k} \subseteq A_{n_{k+1}}$ then $N_{k+1} \geq N_k$.

Since $A_{n_k} \neq A_{n_{k+1}}$ then $N_{k+1} \neq N_k$.

Therefore $N_{k+1} > N_k$, which implies

$$N_{k+1} \geq N_k + 1 \geq k-1+1 = k$$

and completes the induction proving the claim.

Since $A_{n_k} \subseteq D$ then $N_k \leq M$ for all $k \in \mathbb{N}$.

But since $N_k \geq k-1$ this is a contradiction.
Therefore A_n is eventually constant.
Let N be so large that $A_n = A_N$ for $n \geq N$.
Then

$$\bigcup_{n=1}^{\infty} A_n = A_N \in D.$$

Therefore the union of monotone increasing sequences in D are contained in D.

Now let $A_n \in D$ be such that $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$.

Again claim that the sequence is constant.

If is were not constant then there would be a subsequence n_k such that $A_{n_k} \neq A_{n_{k+1}}$ for all $k \in \mathbb{N}$.

Claim $N_k \leq M-k+1$.

We prove this by induction.

Clearly when $k=1$ then $N_1 \leq M-1+1 = M$ since $A_{n_1} \subseteq X$ and there are only M elements in X.

Suppose, for induction, that $N_k \leq M-k+1$.

Since $A_{n_k} \supseteq A_{n_{k+1}}$ then $N_k \geq N_{k+1}$.

Since $A_{n_k} \neq A_{n_{k+1}}$ then $N_k \neq N_{k+1}$.

Therefore $N_k > N_{k+1}$, which implies

$$N_{k+1} \leq N_k - 1 \leq M-k+1-1 = M-(k+1)+1$$

and completes the induction.

Since there can be no fewer than zero elements in a set then $N_k \geq 0$.

But since $N_k \leq M-k+1$ for all $k \in \mathbb{N}$ this is a contradiction. Therefore A_n is eventually constant. Then for some N large enough

$$\bigcap_{n=1}^{\infty} A_n = A_N \in D$$

This proves D is a monotone class.

#2. For two sets A and B show that the following statements are equivalent:

(a) $A \subseteq B$

(b) $A \cup B = B$

(c) $A \cap B = A$.

"(a) $\Rightarrow$ (b)" Suppose $A \subseteq B$ claim $A \cup B = B$.
Obviously $B \subseteq A \cup B$.
Therefore it is enough to show $A \cup B \subseteq B$.

Let $x \in A \cup B$. If $x \in B$ we are done otherwise $x \in A$. Since $A \subseteq B$ this implies $x \in B$. Therefore $A \cup B \subseteq B$.
It follows that $A \cup B = B$.

"(a) $\Rightarrow$ (c)" Suppose $A \subseteq B$ claim $A \cap B = A$.
Obviously $A \cap B \subseteq A$.
Therefore it is enough to show $A \subseteq A \cap B$.

Let $x \in A$. Since $A \subseteq B$ then this implies $x \in B$. Therefore $x \in A \cap B$.
It follows that $A \cap B = A$.

"(b) $\Rightarrow$ (c)" Suppose $A \cup B = B$, claim $A \cap B = A$.

$$A \supseteq A \cap B = A \cap (A \cup B) = (A \cap A) \cup (A \cap B)$$
$$= A \cup (A \cap B) \supseteq A$$

Therefore $A = A \cap B$.

"(c) $\Rightarrow$ (b)" Suppose $A \cap B = A$. Claim $A \cup B = B$.

$$B \subseteq A \cup B = (A \cap B) \cup B = (A \cup B) \cap (B \cup B)$$
$$= (A \cup B) \cap B \subseteq B$$

Therefore $B = A \cup B$.

"(b) $\Rightarrow$ (a)" Suppose $A \cup B = B$. Claim $A \subseteq B$.

$A \subseteq A \cup B = B$.

"(c) $\Rightarrow$ (a)" Suppose $A \cap B = A$. Claim $A \subseteq B$

$B \supseteq A \cap B = A$.

#3. Let A be an uncountable set and B be a countable subset of A. Show $A \sim A \setminus B$.

Let $I \subseteq \mathbb{N}$ be a countable set and

$$b_n : I \to B.$$

be a bijection. Thus $B = \{ b_n : n \in I \}$.

Since A is uncountable and B countable there is a subset $C \subseteq A \setminus B$ that is countably infinite.

Let

$$c_n : \mathbb{N} \to C$$

be a bijection. Thus $C = \{ c_n : n \in \mathbb{N} \}$.

Define $f : A \to A \setminus B$ by

$$f(x) = \begin{cases} c_{2n} & \text{if } x = b_n \text{ for some } n \in I \\ c_{2n+1} & \text{if } x = c_n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$$

Since $C \cap B = \emptyset$ this map is well defined.

<u>Claim f is one-to-one.</u>

Lemma: Suppose $f_n : X_n \to Y_n$ are injective and $X_n \cap X_m = \emptyset$ and $Y_n \cap Y_m = \emptyset$ for $n \neq m$. Let $X = \bigcup X_n$ and $Y = \bigcup Y_n$. Then the piecewise defined function $f : X \to Y$ defined by

$$f(x) = f_n(x) \quad \text{for} \quad x \in X_n$$

is injective.

Proof: Let $a, b \in X$ with $a \neq b$. Then there is n, m such that

$$a \in X_n \quad \text{and} \quad b \in X_m.$$

Case $n = m$: Since $f_n : X_n \to Y_n$ is injective then

$$f(a) = f_n(a) \neq f_n(b) = f(b)$$

implies $f(a) \neq f(b)$.

Case $n \neq m$: Then

$$f(a) = f_n(a) \in Y_n \qquad f(b) = f_m(b) \in Y_m$$

and $Y_n \cap Y_m = \emptyset$ implies $f(a) \neq f(b)$

In both cases $f(a) \neq f(b)$ therefore f is injective.

Returning to problem 3 define

$$X_1 = B \qquad Y_1 = \{c_{2n} : n \in \mathbb{N}\}$$

$$X_2 = C \qquad Y_2 = \{c_{2n+1} : n \in \mathbb{N}\}$$

$$X_3 = A \setminus (B \cup C) \qquad Y_3 = A \setminus (B \cup C)$$

and $f_1 : b_n \to c_{2n}$, $f_2 : c_n \to c_{2n+1}$ and $f_3 : x \to x$.

The functions f_1 and f_2 are injective because $n \to c_n$ is injective. Therefore the hypothesis of the lemma are satisfied and f is injective.

Define $g: A \setminus B \to A$ by $g(x) = x$.

Clearly g is injective.

Thus we have found two injective functions

$$f: A \to A \setminus B \quad \text{and} \quad g: A \setminus B \to A$$

By the Schroeder-Berstein theorem it follows that $A \sim A \setminus B$.

#5 Let $\mathcal{O}$ be a collection of open sets. Then there is a countable subcollection $\{O_1, O_2, \dots\}$ of $\mathcal{O}$ such that $\bigcup_{O\in\mathcal{O}} O = \bigcup O_n$.

Define $U = \bigcup_{O\in\mathcal{O}} O$. For each $x \in U$ there is $O_x \in \mathcal{O}$ such that $x \in O_x$. Since O_x is open there is $r_x > 0$ such that $(x-r, x+r) \subseteq O_x$. Choose rational numbers $\alpha_x, \beta_x \in \mathbb{Q}$ such that $x - r < \alpha_x < x < \beta_x < x + r$. Then

$$x \in (\alpha_x, \beta_x) \subseteq (x-r, x+r) \subseteq O_x \quad \text{for each } x \in U.$$

Define

$$\mathcal{C} = \{(\alpha_x, \beta_x) : x \in U\}.$$

Since $\mathbb{Q} \times \mathbb{Q}$ is countable then $\mathcal{C}$ is contable. Thus

$$\mathcal{C} = \{I_1, I_2, I_3, \dots\}.$$

For each I_n there is some $x_n \in U$, such that $I_n \subseteq O_{x_n}$. Therefore

$$U = \bigcup_{x\in U} \{x\} \subseteq \bigcup_{x\in U} (\alpha_x, \beta_x) = \bigcup I_n \subseteq \bigcup O_{x_n} \subseteq U$$

implies

$$\bigcup_{O\in\mathcal{O}} O = \bigcup O_{x_n}.$$

#6 Suppose $f:[0,1]\to\mathbb{R}$ is continuous and $c\in(0,1)$.

(a) If $f(c)>0$ show there is $h>0$ such that $f(x)>0$ for every x with $|x-c|<h$.

Let $\varepsilon=\frac{1}{2}f(c)$. Then there is $\delta>0$ such that $x\in[0,1]$ and $|x-c|<\delta$ implies $|f(x)-f(c)|<\varepsilon=\frac{1}{2}f(c)$

Choose $h=\min(\delta,c,1-c)$. Then $|x-c|<h$ implies $x\in[0,1]$ so $|f(x)-f(c)|<\varepsilon=\frac{1}{2}f(c)$.

Therefore $|x-c|<h$ implies

$$\begin{aligned} f(x) &= f(c)+f(x)-f(c) \\ &\geq f(c)-|f(x)-f(c)| \\ &> f(c)-\tfrac{1}{2}f(c)=\tfrac{1}{2}f(c)>0 \end{aligned}$$

which proves part (a).

Note that we have actually found $h>0$ such that $f(x)>\frac{1}{2}f(c)$ for all $|x-c|<h$.

(b) If $\int_0^1 (f(x))^2\,dx=0$ then $f\equiv 0$.

Since f is continuous it is enough to show that $f(x)=0$ for every $c\in(0,1)$.

Suppose not. Then there is some $c\in(0,1)$ such that $f(c)\neq 0$.

If $f(c)>0$ then by part (a) we have $h>0$ such that $f(x)>\frac{1}{2}f(c)$ for all $|x-c|<h$.

Let $I=(c-h, c+h)$ and define

$$g(x)=\frac{1}{2}f(c)\chi_I(x)$$

<u>Claim</u> $(f(x))^2 \geq (g(x))^2$ for all $x\in[0,1]$.

<u>Case</u> $x\in I$ then $f(x)>\frac{1}{2}f(c)=g(x)>0$ and therefore $(f(x))^2 \geq (g(x))^2$

<u>Case</u> $x\notin I$ then $(f(x))^2 \geq 0 = (g(x))^2$.

These two cases finish the proof of the claim.

Since $(f(x))^2 \geq (g(x))^2$ for all $x\in[0,1]$ then

$$\int_0^1 (f(x))^2 dx = \int_0^1 (g(x))^2 dx$$

$$= \int_{c-h}^{c+h} \frac{1}{4}(f(c))^2 dx$$

$$= \frac{h}{2}(f(c))^2 > 0$$

Contradicting that $\int_0^1 (f(x))^2 dx = 0$. Therefore there is no point $c\in(0,1)$ such that $f(c)>0$.

If $f(c) < 0$ the repeat the above argument for $\varphi(x) = -f(x)$. It follows that

$$\int_0^1 (f(x))^2 dx = \int_0^1 (\varphi(x))^2 dx > 0$$

again contradicting that $\int_0^1 (f(x))^2 dx = 0$. Therefore there is no point $c \in (0,1)$ such that $f(c) > 0$.

We have show that $f(x) = 0$ for $x \in (0,1)$. Since f is continuous

$$f(0) = \lim_{x \to 0+} f(x) = 0$$

$$f(1) = \lim_{x \to 1-} f(x) = 0$$

and so $f(x) = 0$ for all $x \in [0,1]$ as required.