#4 Let $\sin(x)$ be the sine of x where x is expressed in radians. Define $A_n = \sin n$. Let E be the set of cluster points of A_n. Prove or disprove $E = [-1,1]$.

<u>Proof</u>: Given $n \in \mathbb{N}$ define k_n to be the unique integer such that $n - 2\pi k_n \in [0, 2\pi)$. Define $w_n = 2\pi k_n$. Note that $\sin n = \sin w_n$.

Since w_n is bounded there is a convergent subsequence w_{n_j} such that $w_{n_j} \to w \in [0, 2\pi]$ as $j \to \infty$.

Let $\varepsilon > 0$ and $y \in [-1,1]$. Claim for every $N \in \mathbb{N}$ there is a $n \geq N$ such that $|\sin n - y| < \varepsilon$.

Since w_{n_j} is convergent then it is cauchy. Therefore there is $J \in \mathbb{N}$ such that $i, j \geq J$ implies $|w_{n_j} - w_{n_i}| < \varepsilon$. In particular we have $|w_{n_{J+1}} - w_{n_J}| < \varepsilon$. Thus

$$-\varepsilon < w_{n_{J+1}} - w_{n_J} < \varepsilon$$

$$-\varepsilon < n_{J+1} - n_J - 2\pi(k_{n_{J+1}} - k_{n_J}) < \varepsilon$$

Let $m = n_{J+1} - n_J$. Define $\delta = m - 2\pi(k_{n_{J+1}} - k_{n_J})$. Since n_j is a strictly increasing sequence of natural numbers then $m \in \mathbb{N}$. Moreover since π is irrational then $\delta \neq 0$.

For any $\xi \in \mathbb{R}$ we have

$$\sin(\xi + m) = \sin(\xi + m - 2\pi(k_{n_{J+1}} - k_{n_J})) = \sin(\xi + \delta)$$

Therefore

$$\sin(\xi + m\ell) = \sin(\xi + \delta\ell).$$

We consider two cases: either $\delta > 0$ or $\delta < 0$.

Case $\delta > 0$. Let $L \in \mathbb{N}$ so that $L\delta > 2\pi$. Since $y \in [-1, 1]$ and $\sin x$ is 2π-periodic there is $x \in [N, N+L\delta]$ such that $\sin x = y$. Define $x_\ell = N + \delta \ell$. Choose $\ell \in \{0, 1, \dots, L\}$ so that $|x_\ell - x| < \delta$. Then

$$|\sin(N+\ell m) - y| = |\sin x_\ell - \sin x| \leq |x_\ell - x| < \delta < \varepsilon.$$

Taking $n = N + \ell m$ we have found $n \geq N$ such that

$$|\sin n - y| < \varepsilon.$$

Case $\delta < 0$. Let $L \in \mathbb{N}$ so that $L\delta < -2\pi$. Since $y \in [-1, 1]$ and $\sin x$ is 2π-periodic there is $x \in [N+L\delta, N]$ such that $\sin x = y$. Define $x_\ell = N + \delta \ell$. Choose $\ell \in \{0, 1, \dots, L\}$ so that $|x_\ell - x| < |\delta|$. Then

$$|\sin(N+\ell m) - y| = |\sin x_\ell - \sin x| \leq |x_\ell - x| < |\delta| < \varepsilon$$

Taking $n = N + \ell m$ we have found $n \geq N$ such that

$$|\sin n - y| < \varepsilon$$

Therefore, in either case, we have found $n \geq N$ such that

$$|\sin n - y| < \varepsilon.$$

It follows that y is a cluster point of $\sin n$ and thus $[-1, 1] \subseteq E$. Since the image $\sin \mathbb{N} \subseteq [-1, 1]$ it follows that $E = [-1, 1]$.