Solutions to HW#3

#1. Let x_n and y_n be bounded sequences of real numbers. Prove or disprove the claim that

$$\limsup_{n\to\infty} x_n y_n \le (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n).$$

Counterexample

Let $x_n = y_n = \begin{cases} -1 & \text{if } n \text{ is even} \\ -3 & \text{if } n \text{ is odd}, \end{cases}$

then $\limsup_{n\to\infty} x_n = \limsup_{n\to\infty} y_n = -1$.

Since $x_n y_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 9 & \text{if } n \text{ is odd}, \end{cases}$

then $\limsup_{n\to\infty} x_n y_n = 9$.

Since $9 \le (-1)(-1) = 1$ is a contradiction the claim is false.

2 Let $E \subseteq \mathbb{R}$ and $E' = \{x \in \mathbb{R} : x$ is an accumulation point of $E\}$.
Prove or disprove the claim that $\bar{E} \setminus E'$ is closed.

Counterexample

Let $E = \{ \frac{1}{n} : n \in \mathbb{N} \}$.

then $\bar{E} = \{0\} \cup \{ \frac{1}{n} : n \in \mathbb{N} \}$

and $E' = \{0\}$.

Therefore $\bar{E} \setminus E' = E$ which is not closed.

#3 Let A and B be open subsets of $\mathbb{R}$ such that $\overline{A} = \overline{B}$. Prove or disprove that $A = B$.

<u>Counter example</u>

Let $A = (0,1) \cup (1,2)$

$B = (0,2)$

Then $\overline{A} = [0,2] = \overline{B}$ but $A \neq B$.

#5. Let $D \subseteq \mathbb{R}$. Then the collection $C(D)$ of continuous functions on D is an algebra of functions. That is if $f, g \in C(D)$ and $\alpha \in \mathbb{R}$ then

(a) $f+g \in C(D)$

(b) $\alpha f \in C(D)$

(c) $fg \in C(D)$

Lemma 1. For $a \in D$. If $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$ then $\lim_{x \to a} (f(x) + g(x)) = f(a) + g(a)$

Proof: Let $\varepsilon > 0$.

Choose $\varepsilon_2 = \frac{\varepsilon}{2}$. Since $\lim_{x \to a} f(x) = f(a)$ there is $\delta_2 > 0$ such that

$x \in D$ and $0 < |x-a| < \delta_2$ implies $|f(x) - f(a)| < \varepsilon_2$.

Choose $\varepsilon_3 = \frac{\varepsilon}{2}$. Since $\lim_{x \to a} g(x) = g(a)$ there is $\delta_3 > 0$ such that

$x \in D$ and $0 < |x-a| < \delta_3$ implies $|g(x) - g(a)| < \varepsilon_2$

Choose $\delta = \min(\delta_2, \delta_3)$. Then for $x \in D$ and $0 < |x-a| < \delta$ we have that

$$|f(x) + g(x) - (f(a) + g(a))| \leq |f(x) - f(a)| + |g(x) - g(a)|$$
$$< \varepsilon_1 + \varepsilon_2 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $\lim_{x \to a} (f(x) + g(x)) = f(a) + g(a)$.

By lemma 1 we have show that $f+g \in C(D)$

Lemma 2. For $a \in D$. If $\lim_{x \to a} f(x) = f(a)$ then $\lim_{x \to a} \alpha f(x) = \alpha f(a)$.

Proof. Let $\varepsilon > 0$.

Choose $\varepsilon_2 = \frac{\varepsilon}{1+|\alpha|}$. Since $\lim_{x \to a} f(x) = f(a)$ there is $\delta_2 > 0$ such that

$$x \in D \text{ and } 0 < |x-a| < \delta_2 \text{ implies } |f(x) - f(a)| < \varepsilon_2.$$

Choose $\delta = \delta_2$. Then for $x \in D$ and $0 < |x-a| < \delta$ we have that

$$|\alpha f(x) - \alpha f(a)| = |\alpha| |f(x) - f(a)| < |\alpha| \varepsilon_2$$

$$= \frac{|\alpha|}{1+|\alpha|} \varepsilon < \varepsilon.$$

Therefore $\lim_{x \to a} \alpha f(x) = \alpha f(a)$.

Remark: I took $1+|\alpha|$ in the definition of ε_2 to avoid dividing by zero for the case $\alpha = 0$. Alternatively one could handel the case $\alpha = 0$ directly by noting

$$\lim_{x \to a} \alpha f(x) = \lim_{x \to a} 0 = 0 = \alpha f(a)$$

for any function f when $\alpha = 0$.

By Lemma 2 we have show that $\alpha f \in C(D)$.

Lemma 3. For $a \in D$. If $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$ then $\lim_{x \to a} (f(x)g(x)) = f(a)g(a)$.

Proof: Let $\varepsilon > 0$.

Choose $\varepsilon_2 = \dfrac{\varepsilon}{2(1+|g(a)|)}$. Then since $\lim_{x \to a} f(x) = f(a)$ there is $\delta_2 > 0$ such that $x \in D$ and $0 < |x-a| < \delta_2$ implies $|f(x) - f(a)| < \varepsilon_2$

Choose $\varepsilon_3 = \min\left(1, \dfrac{\varepsilon}{2(1+|f(a)|)}\right)$. Then since $\lim_{x \to a} g(x) = g(a)$ there is $\delta_3 > 0$ such that $x \in D$ and $0 < |x-a| < \delta_3$ implies $|g(x) - g(a)| < \varepsilon_3$.

Let $\delta = \min(\delta_2, \delta_3)$. Then $x \in D$ and $0 < |x-a| < \delta$ implies

$$|f(x)g(x) - f(a)g(a)| \leq |f(x)g(x) - f(a)g(x)| + |f(a)g(x) - f(a)g(a)|$$

$$< |g(x)|\varepsilon_2 + |f(a)|\varepsilon_3 \leq \left(|g(x) - g(a)| + |g(a)|\right)\varepsilon_2 + |f(a)|\varepsilon_3$$

$$\leq (\varepsilon_3 + |g(a)|)\varepsilon_2 + |f(a)|\varepsilon_3 \leq (1 + |g(a)|)\varepsilon_2 + |f(a)|\varepsilon_3$$

$$\leq \frac{\varepsilon}{2} + \frac{|f(a)|}{1+|f(a)|}\frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\lim_{x \to a} (f(x)g(x)) = f(a)g(a)$.

By lemma 3 we have shown $fg \in C(D)$.

Combining lemma 1, 2 and 3 gives that $C(D)$ is an algebra of functions.

#6 Show that a subset of a set of measure zero has measure zero.

Let $A \subseteq \mathbb{R}$ such that $\lambda^*(A) = 0$ and $B \subseteq A$.

Let $\varepsilon > 0$. Since $\lambda^*(A) = 0$ there are intervals I_n such that $A \subseteq \bigcup I_n$ and $\sum_n \ell(I_n) < \varepsilon$.

Since $B \subseteq A$ then $B \subseteq \bigcup I_n$.

Since $\lambda^*(B) = \inf \left\{ \sum_n \ell(I_n) : \begin{array}{l} I_n \text{ are open intervals} \\ \text{and } B \subseteq \bigcup I_n \end{array} \right\}$

it follows that $\lambda^*(B)$ being a lower bound satisfies $\lambda^*(B) < \varepsilon$.

Now $\varepsilon > 0$ arbitrary implies $\lambda^*(B) \leq 0$.

Clearly $\lambda^*(B) \geq 0$.

Therefore $\lambda^*(B) = 0$.