Math 713 Homework 3 Extra Credit Solution

5. Let f be a real valued function defined on the interval $[a,b]$. Let

$$B = \Big\{ c : \lim_{x\to c} f(x) = L \text{ exists but } L \neq f(c) \Big\}$$

be the set of removable discontinuities of f. Prove or disprove the claim that B is a countable set.

Proof: Let

$$P_n = \Big\{ c \in B : \lim_{x\to c} f(x) > f(c) + \frac{1}{n} \Big\}$$

and

$$P_{n,m} = \Big\{ c \in P_n : x \in [a,b] \text{ and } 0 < |x-c| < \frac{1}{m} \text{ implies } f(x) > f(c) + \frac{1}{n} \Big\}.$$

Similarly, let

$$Q_n = \Big\{ c \in B : \lim_{x\to c} f(x) < f(c) - \frac{1}{n} \Big\}$$

and

$$Q_{n,m} = \Big\{ c \in Q_n : x \in [a,b] \text{ and } 0 < |x-c| < \frac{1}{m} \text{ implies } f(x) < f(c) - \frac{1}{n} \Big\}.$$

Then

$$B = \bigcup_{n=1}^{\infty} P_n \cup \bigcup_{n=1}^{\infty} Q_n = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} P_{n,m} \cup \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} Q_{n,m}.$$

Claim that $P_{n,m}$ and $Q_{n,m}$ are finite for each $n, m \in \mathbf{N}$. Suppose $P_{n,m}$ was not finite. Then there would exist a sequence of distinct elements $c_k \in P_{n,m}$. Since c_k lies in the domain of f then it is bounded. Therefore, c_k possesses a convergent subsequence c_{k_j}. Since c_{k_j} converges as $j \to \infty$ then it is Cauchy. Therefore, there is N large enough such that $|c_{k_N} - c_{k_{N+1}}| < 1/m$. Denote $\alpha = c_{k_N}$ and $\beta = c_{k_{N+1}}$. Now, since $\alpha \in P_{n,m}$ then

$$0 < |\beta - \alpha| < 1/m \quad \text{implies} \quad f(\beta) > f(\alpha) + 1/n$$

and since also $\beta \in P_{n,m}$ then

$$0 < |\alpha - \beta| < 1/m \quad \text{implies} \quad f(\alpha) > f(\beta) + 1/n.$$

It follows that $f(\beta) - f(\alpha) > 1/n > -1/n > f(\beta) - f(\alpha)$ which is a contradiction. Therefore $P_{n,m}$ is finite. Similarly, $Q_{n,m}$ is finite.

Since B is the countable union of a countable union of finite sets, then it is countable. This shows that the set of removable discontinuities of f is countable. ////