#1. Let X_n and Y_n be bounded sequences of real numbers. Prove or disprove the claim that

$$\limsup_{n\to\infty}(X_n+Y_n) \le \limsup_{n\to\infty} X_n + \limsup_{n\to\infty} Y_n$$

Proof: Let $L_n = \sup\{X_k : k \ge n\}$ and $M_n = \sup\{Y_k : k \ge n\}$. Since

$$X_n \le \sup\{X_k : k \ge n\} = L_n$$

and

$$Y_n \le \sup\{Y_k : k \ge n\} = M_n$$

then $X_n + Y_n \le L_n + M_n$. It follows that

$$\limsup_{n\to\infty}(X_n+Y_n) \le \limsup_{n\to\infty}(L_n+M_n)$$

Since L_n and M_n are bounded monotone decreasing sequences then L_n+M_n is a bounded monotone decreasing sequence. It follows that

$$\lim_{n\to\infty} L_n, \quad \lim_{n\to\infty} M_n \quad \text{and} \quad \lim_{n\to\infty}(L_n+M_n)$$

exist and therefore

$$\limsup_{n\to\infty}(L_n+M_n) = \lim_{n\to\infty} L_n + \lim_{n\to\infty} M_n$$

$$= \limsup_{n\to\infty} X_n + \limsup_{n\to\infty} Y_n$$

It follows that

$$\limsup_{n\to\infty}(X_n+Y_n) \le \limsup_{n\to\infty} X_n + \limsup_{n\to\infty} Y_n.$$

#2. Let A and B be bounded subsets of real numbers and define

$$A+B=\{a+b : a\in A \text{ and } b\in B\}.$$

Prove of find a counterexample to find the claim $\overline{A}+\overline{B}=\overline{A+B}$.

Proof; We proof inclusion both directions.

"$\subseteq$" Let $x\in \overline{A}+\overline{B}$. Thus $x=a+b$ where $a\in\overline{A}$ and $b\in\overline{B}$. It follows that there are sequences $a_n\in A$ and $b_n\in B$ such that $a_n\to a$ and $b_n\to b$ as $n\to\infty$.

Define $x_n=a_n+b_n$.

Then $x_n\in A+B$ for each n and

$$\lim_{n\to\infty} x_n=\lim_{n\to\infty}(a_n+b_n)=\lim_{n\to\infty} a_n+\lim_{n\to\infty} b_n=a+b=x$$

shows that $x\in\overline{A+B}$.

"$\supseteq$" Let $x\in\overline{A+B}$. Thus there is a sequence $x_n\in A+B$ such that $x_n\to x$ as $n\to\infty$. Since $x_n\in A+B$ there is $a_n\in A$ and $b_n\in B$ such that $x_n=a_n+b_n$. Since A is bounded then a_n is a bounded sequence. Therefore there is a convergent subsequence a_{n_j} and some element $a\in\mathbb{R}$ such that $a_{n_j}\to a$ as $j\to\infty$

Since B is bounded then b_{n_j} is a bounded subsequence of b_n. Therefore there is a convergent sub-subsequence $b_{n_{j_k}}$ and some element $b\in\mathbb{R}$ so $b_{n_{j_k}}\to b$ as $k\to\infty$.

Since a subsequence of a convergent sequence also converges, we have $a_{n_{j_k}} \to a$ as $k \to \infty$.

Thus $a_{n_{j_K}} \in A$ and $b_{n_{j_k}} \in B$ and

$$\lim_{K\to\infty} a_{n_{j_K}} = a \quad \text{and} \quad \lim_{K\to\infty} b_{n_{j_K}} = b$$

implies $a \in \bar{A}$ and $b \in \bar{B}$.

Since

$$x = \lim_{n\to\infty} x_n = \lim_{K\to\infty} x_{n_{j_K}}$$

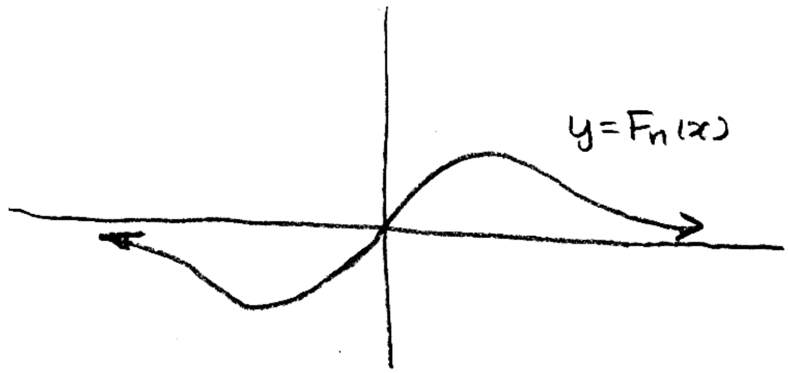
$$= \lim_{K\to\infty} (a_{n_{j_K}} + b_{n_{j_K}})$$

$$= \lim_{K\to\infty} a_{n_{j_K}} + \lim_{K\to\infty} b_{n_{j_K}} = a+b$$

It follows that $x \in \bar{A} + \bar{B}$.

#3 Let $F_n(x) = \dfrac{x}{1+nx^2}$. Prove or disprove the claim that the sequence F_n converges uniformly on $\mathbb{R}$.

Proof: For each n the graph of F_n looks like



Since $F_n(-x) = -F_n(x)$ symmetry implies

$$|F_n(x)| \leq \max\{F_n(x) : x \in \mathbb{R}\}.$$

Use Calculus to compute this maximum.

$$F_n'(x) = \frac{1+nx^2 - x(2nx)}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2} = 0$$

implies $x = \pm\frac{1}{\sqrt{n}}$. The maximum is at $x = \frac{1}{\sqrt{n}}$ and thus

$$|F_n(x)| \leq F_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1/\sqrt{n}}{1+n(1/\sqrt{n})^2} = \frac{1}{2\sqrt{n}}$$

for every $x \in \mathbb{R}$.

To see $F_n \to 0$ uniformly, let $\varepsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ so large that $\frac{1}{2\sqrt{N}} < \varepsilon$.

Then for $n \geq N$ and every $x \in \mathbb{R}$ we have

$$|F_n(x) - 0| = |F_n(x)| \leq \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} < \varepsilon .$$

#5. Prove that every monotone function is Borel measurable.

If f is monotone decreasing then $-f$ is monotone increasing. Since $\hat{C}$ is an algebra of functions then showing $-f \in \hat{C}$ implies that $f \in \hat{C}$. Therefore we may assume, without loss of generality, that f is a monotone increasing function.

By lemma 3.5(b) it is enough to show that

$$f^{-1}((a,\infty)) \in \mathcal{B} \quad \text{for every} \quad a \in \mathbb{R}.$$

Let $a \in \mathbb{R}$ and define $\alpha = \inf\{x : f(x) > a\}$.

Claim $(\alpha, \infty) \subseteq f^{-1}((a,\infty))$.

Let $x \in (\alpha, \infty)$. Since $x > \alpha$ then x is not a lower bound for $f^{-1}((a,\infty))$. Therefore there is $x_0 \in f^{-1}((a,\infty))$ such that $x_0 < x$. Since f is increasing then $f(x) \geq f(x_0) > a$. Therefore it follows that $x \in f^{-1}((a,\infty))$.

If $\alpha = -\infty$ then $(-\infty,\infty) \subseteq f^{-1}((a,\infty))$ implies $f^{-1}((a,\infty)) = \mathbb{R} \in \mathcal{B}$.
If $\alpha < \infty$ then $x \in f^{-1}((a,\infty))$ implies $x \geq \alpha$ or $x \in [\alpha, \infty)$.
Therefore $(\alpha,\infty) \subseteq f^{-1}((a,\infty)) \subseteq [\alpha,\infty)$ and it follows that either $f^{-1}((a,\infty)) = (\alpha,\infty)$ or $f^{-1}((a,\infty)) = [\alpha,\infty)$. Since $\mathcal{B}$ contains all the open and closed sets, then in both cases we obtain that $f^{-1}((a,\infty)) \in \mathcal{B}$.

#6 Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable at each point of $\mathbb{R}$ and $|f'(x)| \le 1$ for each $x \in \mathbb{R}$. Prove that for each $A \subseteq \mathbb{R}$ that $\lambda^*(f(A)) \le \lambda^*(A)$.

<u>Lemma</u> If $I = (a,b)$ then $\lambda^*(f(I)) \le \lambda^*(I)$.

Proof: Let $\alpha = \inf f(I)$ and $\beta = \sup f(I)$, Then $f(I) \subseteq [\alpha, \beta]$. By Proposition 3.1(c) $\lambda^*(f(I)) \le \lambda^*([\alpha,\beta])$. By Proposition 3.2 $\lambda^*([\alpha,\beta]) = \beta - \alpha$.

Let $\varepsilon > 0$ be arbitrary. Since $\alpha + \varepsilon/2$ is not a lower bound of $f(I)$ there is $x_1 \in I$ such that $f(x_1) < \alpha + \varepsilon/2$. Since $\beta - \varepsilon/2$ is not an upper bound of $f(I)$ there is $x_2 \in I$ such that $f(x_2) > \beta - \varepsilon/2$. Therefore

$$\lambda^*(f(I)) \le \beta - \alpha < f(x_2) - f(x_1) + \varepsilon .$$

By the mean value theorem there is ξ between x_1 and x_2 such that $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$. Hence

$$|f(x_2) - f(x_1)| = |f'(\xi)||x_2 - x_1| \le |x_2 - x_1| < b - a$$

Since $x_1, x_2 \in I = (a,b)$. It follows that

$$\lambda^*(f(I)) \le b - a + \varepsilon = \ell(I) + \varepsilon = \lambda^*(I) + \varepsilon$$

Since ε is arbitrary, then

$$\lambda^*(f(I)) \le \lambda^*(I).$$

This proves the lemma.

Now we use the lemma to prove that $\lambda^*(f(A)) \leq \lambda^*(A)$ for every $A \subseteq \mathbb{R}$.

Given $A \subseteq \mathbb{R}$, let $\varepsilon > 0$ be arbitrary. Let I_n be a sequence of open intervals such that $A \subseteq \bigcup_n I_n$ and

$$\sum_n \ell(I_n) < \lambda^*(A) + \varepsilon.$$

Then $f(A) \subseteq f(\bigcup_n I_n)$ and proposition 3.1 (c) and (e) implies

$$\lambda^*(f(A)) \leq \lambda^*\left(f\left(\bigcup_n I_n\right)\right) = \lambda^*\left(\bigcup_n f(I_n)\right)$$

$$\leq \sum_n \lambda^*(f(I_n)),$$

Now the lemma implies $\lambda^*(f(I_n)) \leq \lambda^*(I_n)$. Thus

$$\lambda^*(f(A)) \leq \sum_n \lambda^*(I_n) = \sum_n \ell(I_n) < \lambda^*(A) + \varepsilon.$$

Since ε is arbitrary we obtain

$$\lambda^*(f(A)) \leq \lambda^*(A).$$