Math 713 Homework 6 Solutions Dec 10 '10

#1. Let g_n be any sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ and g be defined by

$$g(x) = \inf\{g_n(x) : n = 1, 2, 3, \ldots\}.$$

Prove or disprove the claim that

$$g^{-1}((a,\infty)) = \bigcap_{n=1}^{\infty} g_n^{-1}((a,\infty)).$$

This claim is false.

Consider $g_n(x) = \frac{1}{n}$. Then $g(x) = \inf\{\frac{1}{n} : n = 1, 2, 3 \ldots\} = 0$.
Thus

$$g^{-1}((0,\infty)) = \{x \in \mathbb{R} ; g(x) > 0\} = \{x \in \mathbb{R} : 0 > 0\} = \emptyset$$

whereas

$$g_n^{-1}((0,\infty)) = \{x \in \mathbb{R} : g_n(x) > 0\} = \{x \in \mathbb{R} : \tfrac{1}{n} > 0\} = \mathbb{R}.$$

Therefore

$$\bigcap_{n=1}^{\infty} g_n^{-1}((0,\infty)) = \mathbb{R} \neq \emptyset = g^{-1}((0,\infty))$$

#2 Let $E \subseteq \mathbb{R}$. Suppose for every $\varepsilon > 0$ there is a Borel measurable function such that $F \subseteq E$ and $\lambda^*(E \setminus F) < \varepsilon$. Prove or disprove the claim that E is Borel measurable.

The claim is false.

Let E be the set H given by problem 3.50 such that $\lambda(E) = 0$ but $E \notin \mathcal{B}$. Thus for $\varepsilon > 0$ we can choose $F = \emptyset$ so that $F \subseteq E$ and $\lambda^*(E \setminus F) = \lambda^*(E) = 0$.

However $E \notin \mathcal{B}$.

3.50 Prove there exists a set $H \in \mathcal{M} \setminus \mathcal{B}$ such that $\lambda(H) = 0$.

(a) If $C \in \mathcal{M}$ and $x \in \mathbb{R}$ then $C+x \in \mathcal{M}$ and $\lambda(C+x) = \lambda(C)$.

Claim $W \cap (C+x) = ((W-x) \cap C) + x$.

"$\subseteq$" $z \in W \cap (C+x)$ implies $z \in W$ and $z \in C+x$.

$z \in W$ implies $z - x \in W - x$.

$z \in C+x$ implies $z - x \in C$.

Thus $z - x \in (W-x) \cap C$.

Therefore $z \in ((W-x) \cap C) + x$.

"$\supseteq$" $z \in ((W-x) \cap C) + x$ implies $z - x \in (W-x) \cap C$.

Thus $z - x \in W - x$ and $z - x \in C$.

$z - x \in W - x$ implies $z \in W$

$z - x \in C$ implies $z \in C + x$.

Therefore $z \in W \cap (C+x)$.

Claim $W \cap (C+x)^c = ((W-x) \cap C^c) + x$

"$\subseteq$" $z \in W \cap (C+x)^c$ implies $z \in W$ and $z \notin C+x$.

$z \in W$ implies $z - x \in W - x$

$z \notin C+x$ implies $z - x \notin C$.

Thus $z - x \in (W-x) \cap C^c$

Therefore $z \in ((W-x) \cap C^c) + x$.

"$\supseteq$" $z \in ((W-x) \cap C^c) + x$ implies $z - x \in (W-x) \cap C^c$.

Thus $z - x \in W - x$ and $z - x \notin C$

$z - x \in W - x$ implies $z \in W$

$z - x \notin C$ implies $z \notin C + x$

Therefore $z \in W \cap (C+x)^c$.

To show that $C+x \in M$ we need to show it satisfies the cartheodory criterion. Since $C \in M$ then

$$\lambda^*(W) = \lambda^*(W \cap C) + \lambda^*(W \cap C^c) \text{ for every } W \subseteq \mathbb{R}.$$

Since this holds for every subset then it also holds for $W-x$.

$$\lambda^*(W-x) = \lambda^*((W-x) \cap C) + \lambda^*((W-x) \cap C^c) \text{ for every } W \subseteq \mathbb{R}.$$

By Proposition 3.1 we have $\lambda^*(W-x) = \lambda^*(W)$. By the claims on the previous page $(W-x) \cap C = W \cap (C+x)$ and $(W-x) \cap C^c = W \cap (C+x)^c$. It follows that

$$\lambda^*(W) = \lambda^*(W \cap (C+x)) + \lambda^*(W \cap (C+x)^c) \text{ for every } W \subseteq \mathbb{R}.$$

Thus $C+x$ satisfies the cartheodory condition. Thus $C+x \in M$. Again Proposition 3.1 implies

$$\lambda(C+x) = \lambda^*(C+x) = \lambda^*(C) = \lambda(C).$$

(b) Let S be the set defined in Lemma 3.12. If $C \in M$ and $C \subseteq S$, then $\lambda(C) = 0$.

Since $\{S+r : r \in (-1,1) \cap \mathbb{Q}\}$ is a disjoint collection of subsets, then $\{C+r : r \in (-1,1) \cap \mathbb{Q}\}$ is a disjoint collection of measurable subsets since $C+r$ is measurable by part (a).

Moreover $\bigcup_{r \in (-1,1) \cap \mathbb{Q}} C+r \subseteq \bigcup_{r \in (-1,1) \cap \mathbb{Q}} S+r \subseteq [-1,2]$ implies

$$\lambda\Big(\bigcup_{r \in (-1,1) \cap \mathbb{Q}} C+r\Big) = \sum_{r \in (-1,1) \cap \mathbb{Q}} \lambda(C+r) = \sum_{r \in (-1,1) \cap \mathbb{Q}} \lambda(C) \leq \lambda([-1,2]) = 3$$

In particular $\sum_{r \in (-1,1) \cap \mathbb{Q}} \lambda(C) < \infty$.

This implies $\lambda(C) = 0$.

(c) If $D \subseteq \mathbb{R}$ and $\lambda^*(D) > 0$, then there is a non-measurable subset of D.

Define $D_r = D \cap (S + r)$ for $r \in \mathbb{Q}$.

Since $D_r - r = (D - r) \cap S$ by the claim in part (a), then $D_r \in \mathcal{M}$ implies $D_r - r \in \mathcal{M}$ and $D_r - r \subseteq S$. It would then follow from part (b) that $\lambda(D_r - r) = 0$. Therefore $D_r \in \mathcal{M}$ implies $\lambda(D_r) = 0$.

Since $D_r \subseteq S + r$, then $\{D_r : r \in \mathbb{Q}\}$ is a disjoint collection of subsets. Suppose, for contradiction that, $D_r \in \mathcal{M}$ for every r. Then $\lambda(D_r) = 0$ for every r. Therefore,

$$D = \bigcup_{r \in \mathbb{Q}} D_r$$

would imply that

$$\lambda^*(D) \leq \sum_{r \in \mathbb{Q}} \lambda^*(D_r) = \sum_{r \in \mathbb{Q}} \lambda(D_r) = 0$$

This contradicts that $\lambda^*(D) > 0$. It follows that for at least some $r \in \mathbb{Q}$ that $D_r \notin \mathcal{M}$.

(d) Define $\varphi : [0,1] \to \mathbb{R}$ by $\varphi(x) = x + \psi(x)$ where ψ denotes the Cantor function. Claim that φ is a strictly increasing function which maps $[0,1]$ onto $[0,2]$.

Since $\psi : [0,1] \to [0,1]$ is onto then $\varphi : [0,1] \to [0,2]$ is onto. To show φ is strictly increasing we use the fact that ψ is non-decreasing.

Let $x_1 < x_2$. Then

$$\varphi(x_1) = x_1 + \psi(x_1) < x_2 + \psi(x_1) \leq x_2 + \psi(x_2) = \varphi(x_2)$$

Therefore φ is strictly increasing and onto.

(c) The function $g = \varphi^{-1}$ is continuous and hence Borel measurable

Since $g : [0,2] \to [0,1]$ is monotone and onto then

$$h : \mathbb{R} \to \mathbb{R}$$

$$h : (x) = \begin{cases} g(x) & \text{if } x \in [0,2] \\ g(0) - x & \text{if } x < 0 \\ g(2) + x - 2 & \text{if } x > 2 \end{cases}$$

is monotone and onto $\mathbb{R}$. By the first lemma proved in the lecture notes from November 10, we have that G is continuous. It immediately follows that $g = h|_{[0,2]}$ is continuous.

By definition every continuous function is Borel measurable.

(f) The function φ maps the Cantor set P onto a set A with $\lambda(A) = 1$.

By definition

$$\varphi(x) = \begin{cases} x + f(x) & \text{if } x \in P \\ x + f(a_n) & \text{if } x \in (a_n, b_n) \end{cases}$$

where f is the function used in definition of the Cantor function and (a_n, b_n) are the unique disjoint open intervals such that $G = \bigcup_{n=1}^{\infty} (a_n, b_n)$ given by the structure theorem of the open sets of the real line where $G = [0,1] \setminus P$.

<u>Claim</u> $\lambda(\varphi(G)) = 1$. Recall that G is comprised of one open interval of length $1/3$, two intervals of length $1/3^2$ and in general 2^{n-1} intervals of length $1/3^n$.

Now

$$\lambda(\varphi(G)) = \lambda\left(\varphi\left(\bigcup_{n=1}^{\infty}(a_n, b_n)\right)\right)$$

$$= \sum_{n=1}^{\infty} \lambda\left(\{x + f(a_n) : x \in (a_n, b_n)\}\right)$$

$$= \sum_{n=1}^{\infty} \lambda\left((a_n, b_n) + f(a_n)\right) = \sum_{n=1}^{\infty} \lambda((a_n, b_n))$$

$$= \lambda\left(\bigcup_{n=1}^{\infty}(a_n, b_n)\right) = \lambda(G) = 1$$

It follows, since φ is one-to-one, that

$$\lambda(\varphi(P)) = \lambda(\varphi([0,1])) - \lambda(\varphi(G))$$

$$= \lambda([0,2]) - 1 = 2 - 1 = 1,$$

(g) Let $E \subseteq A$ with $E \notin M$. Then $\varphi^{-1}(E) \in M$ but $\varphi^{-1}(E) \notin \mathcal{B}$.
Recall $A = \varphi(P)$. Since $\lambda(A) > 0$ then E exists.
Now $E \subseteq \varphi(P)$ implies $\varphi^{-1}(E) \subseteq \varphi^{-1}(\varphi(P)) = P$.
Therefore $\lambda^*(\varphi^{-1}(E)) \leq \lambda^*(P) = 0$. Thus $\varphi^{-1}(E) \in M$.

Suppose, for contradiction, that $\varphi^{-1}(E) \in \mathcal{B}$. Since φ^{-1} is continuous then it's inverse takes $\mathcal{B}$ into $\mathcal{B}$. Therefore

$$E = \varphi(\varphi^{-1}(E)) = (\varphi^{-1})^{-1}(\varphi^{-1}(E)) \in \mathcal{B}$$

However, this is a contradiction, since $E \notin M$.
Therefore $\varphi^{-1}(E) \notin \mathcal{B}$.

Thus taking $H = g^{-1}(E)$ yields a set $H \in M$ such that $H \notin \mathcal{B}$.

#3 Suppose that $A, B \in M$ are such that $A \subsetneq B$ and $\lambda(A) < \infty$. Show that $\lambda(B \setminus A) = \lambda(B) - \lambda(A)$.

$A, B \in M$ therefore $B \setminus A \in M$.

Since $B = B \setminus A \cup A$ and $B \setminus A$ and A are disjoint then

$$\lambda(B) = \lambda(B \setminus A) + \lambda(A).$$

Since $\lambda(A) < \infty$ we can subtract it from both sides of the above equality to obtain

$$\lambda(B \setminus A) = \lambda(B) - \lambda(A).$$

#4 Suppose $E_n \in \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq \cdots$. Also suppose $\lambda(E_1) < \infty$.
Prove that
$$\lambda\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n)$$

Define $A_n = E_1 \setminus E_n$. Thus $A_1 \subseteq A_2 \subseteq \cdots$ and $A_n \in \mathcal{M}$. From Theorem 3.13 it follows that

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_n)$$

Since $\lambda(E_1) < \infty$ then $\lambda(E_n) < \infty$ for all $n \in \infty$.

Now the previous problem implies

$$\lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \lambda\left(\bigcup_{n=1}^{\infty} E_1 \setminus E_n\right) = \lambda\left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right)$$
$$= \lambda(E_1) - \lambda\left(\bigcap_{n=1}^{\infty} E_n\right)$$

and similarly

$$\lim_{n\to\infty} \lambda(A_n) = \lim_{n\to\infty} \lambda(E_1 \setminus E_n) = \lim_{n\to\infty} \left(\lambda(E_1) - \lambda(E_n)\right)$$
$$= \lambda(E_1) - \lim_{n\to\infty} \lambda(E_n).$$

Again since $\lambda(E_1) < \infty$ this implies

$$\lambda\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \lambda(E_n).$$

#6 Suppose E is Lebesgue measurable and $\lambda(E) > 0$. Prove or disprove the claim that there is an open interval I such that $\lambda(E \cap I) > \lambda(I)/2$.

For contradiction, suppose $\lambda(E \cap I) \leq \lambda(I)/2$ for every open interval I.

Let $\varepsilon = \lambda(E) > 0$.

By definition of $\lambda(E)$ there exists I_n open intervals such that $\sum \ell(I_n) < \lambda(E) + \varepsilon$. Now

$$\sum \ell(I_n) = \sum \lambda(I_n) \geq 2 \sum \lambda(E \cap I_n)$$
$$\geq 2\lambda\left(\bigcup_{n=1}^{\infty} E \cap I_n\right) = 2\lambda(E)$$

implies

$$2\lambda(E) < \lambda(E) + \varepsilon = 2\lambda(E)$$

which is a contradiction.

Therefore there is some open interval I such that

$$\lambda(E \cap I) > \lambda(I)/2.$$