Math 713 Summary for Aug 23, '10

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Office hours: MW 11-12 in DMS 238.

There will be a 5 minute break.

Lebesgue theory is motivated by taking limits under integrals when obtaining solutions to differential equations as the limit of approximate solutions.

Calculus review

$$\int_{-\infty}^{\infty} x e^{-x^2}\, dx = 0$$

$$\int_{-\infty}^{\infty} |x| e^{-x^2}\, dx = 1$$

$$\int_{-\infty}^{\infty} |x| e^{-nx^2}\, dx = \frac{1}{n}$$

Let $f_n(x) = n|x| e^{-nx^2}$

$$\int_{-\infty}^{\infty} f_n(x)\, dx = 1$$

So $\lim\limits_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx = 1.$

What is $\lim\limits_{n\to\infty} f_n(x)$?

If $x \neq 0$ then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n|x|}{e^{nx^2}}$$

$$= \lim_{n\to\infty} \frac{|x|}{x^2 e^{nx^2}} = 0$$

Thus the integral of the limit is zero whereas the limit of the integral is one for this sequence of functions.

Therefore

$$\lim_{n\to\infty} f_n(x) = \begin{Bmatrix} 0 \text{ if } x=0 \\ 0 \text{ if } x\neq 0 \end{Bmatrix} = 0.$$

How do you prove the fundamental theorem of Calculus?

How long ago did you prove it?

What is a theorem used to prove it?

Can you state the mean value theorem?

Mean Value Theorem: If f is continuous on $[a,b]$ and differentiable on (a,b), then there is c in (a,b) such that

$$f(b) - f(a) = f'(c)(b-a)$$

The mean value theorem is stated as Theorem 5.5 in Dangello and Seyfried.

Notation from chapter 1.

Book	Blackboard
$\mathbf{R}$	$\mathbb{R}$
$\mathbf{Q}$	$\mathbb{Q}$
$\mathbf{Z}$	$\mathbb{Z}$
$\mathbf{N}$	$\mathbb{N}$

DeMorgan's Laws

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

come from the logical equivalences

not(p or q) $\iff$ (not p) and (not q)

not (p and q) $\iff$ (not p) or (not q)

which can be proven by plugging in all possible truth values for p and q from a truth table

If you have not heard about Russell's paradox, look it up.

p	q	p or q	not(p or q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

same

p	q	not p	not q	(not p) and (not q)
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Image and Inverse image

$$f^{-1}(B) = \{ x : f(x) \in B \}$$

$$f(B) = \{ f(x) : x \in B \}$$

Give an example so

$$f\Big(\bigcap_{i\in I} A_i\Big) \subseteq \bigcap_{i\in I} f(A_i).$$

and that the set of the left is strictly contained in the set on the right.

For example

$$f(x) = 3, \qquad A_i = \left(0, \tfrac{1}{i}\right)$$

then

$$\bigcap_{i\in I} A_i = \emptyset$$

$$f(A_i) = \left\{ f(x) : x \in \left(0, \tfrac{1}{i}\right) \right\} = \{3\}$$

So $$\bigcap_{i\in I} f(A_i) = \{3\}$$

$$f\Big(\bigcap_{i\in I} A_i\Big) = f(\emptyset) = \emptyset$$

The Lebesgue integral is an extension of the Riemann integral in the sense that it is consistent. For example addition on fractions is an extension of addiotion on the natural numbers:

$$\frac{2}{2} + \frac{3}{3} = \frac{2 \cdot 3 + 3 \cdot 2}{2 \cdot 3} = \frac{12}{6}$$

is the same as

$$1 + 1 = 2$$

Even though many functions are able to be integrated in the Lebeggue sense not all. To find a function that is not Lebesgue integrable requires the <u>Axiom of Choice</u>

suppose $\mathcal{C}$ is a collection of non-empty sets. Then there exists a function $f: \mathcal{C} \to \bigcup_{A \in \mathcal{C}} A$ such that $f(A) \in A$ for each $A \in \mathcal{C}$.