Math 713 Summary Aug 25

Equivalence of sets.

$$A \sim B$$

means there is a 1-to-1 function from A onto B. That is there is a bijection between A and B.

<u>Proposition 1.7</u> A nonempty set A is countable if it is the range of an infinite sequence, that is there is a function $f: \mathbb{N} \to A$ which is onto.

<u>Idea</u> build a bijection one step at a time

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$f(n)$	1	2	$\frac{1}{2}$	$\frac{1}{3}$	$\cancel{2}$	$\cancel{\frac{1}{2}}$	$\frac{1}{4}$	$\cancel{\frac{1}{3}}$	$\cancel{\frac{1}{3}}$	$\frac{2}{3}$	17	18	$\cancel{\frac{1}{2}}$

$h(m)$	1	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{2}{3}$	17	18	...
m	1	2	3	4	5	6	7	8	...

Either the construction stops because there were only finite many elements in A to begin with or there is always a way to define $h(m+1)$ given $h(1)$, $h(2), \ldots, h(m)$.

The proof presented in class is difficult to explain on a paper, but is exactly the idea from page 22 of the text.

please read the text to see how this idea is properly explained in a written form.

Homework assignment is made and questions are on the website

http://fractal.math.unr.edu/~ejolson/181/

Claim that $\mathbb{N} \sim \mathbb{Q}$.

First $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$. We need a bijection, $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. For example:

$$\begin{array}{llll}
1\,(1,1) & 3\,(1,2) & 4\,(1,3) & 10\,(1,4) \\
2\,(2,1) & 5\,(2,2) & 9\,(2,3) & (2,4) \\
6\,(3,1) & 8\,(3,2) & (3,3) & \\
7\,(4,1) & & &
\end{array}$$

anything that fills in the top left corner as you count $1, 2, 3, \ldots$ will work.

Now define $g: \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ by

$$g(p,q) = \begin{cases} \dfrac{p}{2q} & \text{if } p \text{ even} \\[2ex] -\dfrac{(p-1)}{2q} & \text{if } p \text{ odd} \end{cases}$$

to obtain $h: \mathbb{N} \to \mathbb{Q}$ onto given by

$h(n) = g \circ f(n)$

It follows that

$\mathbb{Q}$ is countable

Since $\mathbb{Q}$ is obviously countably infinite we have that $\mathbb{N} \sim \mathbb{Q}$.

For next time read the proof of Theorem 1.1 and try to figure out the construction used in the proof of the Schröder-Bernstein theorem on page 26.

Calculus review:

How to prove the mean value theorem.

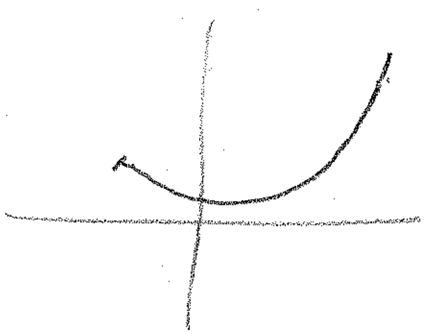
Roll's theorem

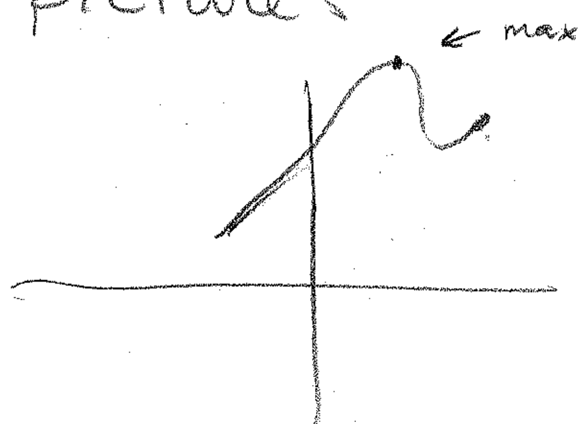
Let $f : [a,b] \to \mathbb{R}$ be continuous, differentiable on (a,b) with $f(a) = f(b)$. Then there is $c \in (a,b)$ such that $f'(c) = 0$.

Extreme value theorem

Let $f : [a,b] \to \mathbb{R}$ be continuous, differentiable on (a,b). Then the maximum of f is either $f(a)$ or $f(b)$ or at some point $c \in (a,b)$ where $f'(c) = 0$.

<u>Proof</u>: Draw pictures





Put pictures into words:

If $f(a)$ or $f(b)$ is maximum, we are done. Otherwise the maximum is at some $c \in (a,b)$. Now $f(c+h) \leq f(c)$ for any h, so $c+h \in [a,b]$. It follows that.

Case $h<0$, $\dfrac{f(c+h)-f(c)}{h} \geq 0$

Case $h>0$ $\dfrac{f(c+h)-f(c)}{h} \leq 0$

Therefore

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

and

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0.$$

It follows that

$$0 \le f'(c) \le 0$$

and so $f'(c) = 0$.

Considering $-f$ we have the same result for the minimum.

To prove Roll's theorem note that if the maximum and minimum occur at either a or b then the function must be a horizontal line. In this case $f'(c) = 0$ for every $c \in (a,b)$.

In the case the maximum or minimum occurs at $c \in (a,b)$ then the extreme value theorem states that $f'(c) = 0$.

The proof of the mean value theorem may be obtained from Rolls theorem by a change of coördinates. Try this at home.