Math 713 Summary Aug 30

Proof of Schroeder-Bernstein Theorem

This is homework # 1.36 on page 26 of the text and related to problems 3 and 4 on the homework assignment.

Suppose A, B are sets and there are one-to-one functions $f: A \to B$ and $g: B \to A$. Then $A \sim B$. That is there exists a bijection $h: A \to B$.

Lemma A: Define $\tau(E) = g(f(E)^c)^c$ for $E \subseteq A$. Show that $E \subseteq F \subseteq A$ implies $\tau(E) \subseteq \tau(F)$.

A little matter of notation...

$A \subset B$ means $A \subseteq B$ or $A = B$

BUT

$a < b$ means $a \leq b$ and $a \neq b$.

In or text $A \subset B$ means

for every $x \in A$ then $x \in B$.

That is $\subset$ and $\subseteq$ have exactly the same meaning.

Note $A \subsetneq B$ means

not $A \subseteq B$

which is different than

$A \subseteq B$ and $A \neq B$.

Proof of Lemma A

Since $E \subseteq F$

$$f(E) \subseteq f(F)$$

$$f(E)^c \supseteq f(F)^c$$

$$g(f(E)^c) \supseteq g(f(F)^c),$$

Then $g(f(E)^c)^c \subseteq g(f(F)^c)^c$

There for $\tau(E) \subseteq \tau(F)$.

Lemma B: Define $\mathcal{C} = \{E \subseteq A : E \subseteq \tau(E)\}$. and $G = \bigcup_{E \in \mathcal{C}} E$. Then $\tau(G) = G$.

First, for $E \in \mathcal{C}$ then $E \subseteq G$. Thus $\tau(E) \subseteq \tau(G)$ by lemma A. Therefore

$$G = \bigcup_{E \in \mathcal{C}} E \subseteq \bigcup_{E \in \mathcal{C}} \tau(E) \subseteq \bigcup_{E \in \mathcal{C}} \tau(G) = \tau(G)$$

shows $G \subseteq \tau(G)$.

To show the inclusion $\tau(G) \subseteq G$ define $F = G \cup \tau(G)$.

Then

$$G \subseteq F \quad \text{and} \quad \tau(G) \subseteq F$$

So by Lemma A

$$\tau(G) \subseteq \tau(F) \quad \text{and} \quad \tau(\tau(G)) \subseteq \tau(F)$$

Now $G \subseteq \tau(G)$ from as shown in the first part of the proof. Thus

$$\tau(G) \subseteq \tau(\tau(G))$$

It follows that

$$G \subseteq \tau(G) \subseteq \tau(F) \quad \text{and} \quad \tau(G) \subseteq \tau(\tau(G)) \subseteq \tau(F)$$

Therefore

$$F = G \cup \tau(G) \subseteq \tau(F) \cup \tau(F) = \tau(F)$$

implies $F \in \mathcal{C}$. But then

$$G \supseteq F = G \cup \tau(G)$$

shows $\tau(G) \subseteq G$.

Proof of theorem

Define $h: A \to B$

$$h(x) = \begin{cases} f(x) & \text{if } x \in G \\ g^{-1}(x) & \text{if } x \notin G. \end{cases}$$

Claim ① h is well defined
② h is 1-to-1
③ h is onto B.

① Clearly if $x \in G$ there is no problem. Need to show $g^{-1}(x)$ makes sense if $x \notin G$. In this case

$$x \in G^c = \tau(G)^c = \left(g(f(G)^c)^c\right)^c = g(f(G)^c) \subseteq g(B)$$

So x is in the range of g. Since g is one-to-one then $g^{-1}(x)$ makes sense and h is well defined.

③ Claim h is onto.

(This is not done in class)

Let $y \in B$. If there exists $x \in G$ such that $f(x) = y$ we are done otherwise $y \notin f(G)$. Thus

$$\{y\} \subseteq f(G)^c$$

$$g(\{y\}) \subseteq g(f(G)^c)$$

$$g(\{y\})^c \supseteq g(f(G)^c)^c = \tau(G) = G$$

Thus

$$g(\{y\}) \subseteq G^c$$

Let $x = g(y)$. Then $x \in G^c$ and

$$h(x) = g^{-1}(x) = g^{-1} \circ g(y) = y$$

which shows that h is onto.

(2) Claim h is 1-to-1. Let $x \neq y$

Case $x, y \in G$, then f one-to-one implies
$h(x) = f(x) \neq f(y) = h(y)$

Case $x, y \in G^c$, then g^{-1} one to one implies
$h(x) = g^{-1}(x) \neq g^{-1}(y) = h(y)$

Case $x \in G$ and $y \in G^c$. For contradiction assume $f(x) = h(x) = h(y) = g^{-1}(y)$.

Thus $f(\{x\}) = g^{-1}(\{y\})$

$f(\{x\})^c = g^{-1}(\{y\})^c = g^{-1}(\{y\}^c)$

by homework problem 1. Thus

$$g(f(\{x\})^c) = g(g^{-1}(\{y\}^c)) = \{y\}^c$$

and so

$$G = \mathcal{T}(G) = g(f(\{x\})^c)^c = \{y\}$$

Shows $y \in G$ contradicting $y \in G^c$.
It follows that $h(x) \neq h(y)$

Section 1.4

Definition of

<u>Algebra of sets</u>: A nonempty collection $\mathcal{A}_0$ of subsets of Ω is called an algebra if;

① $A \in \mathcal{A}_0 \Rightarrow A^c \in \mathcal{A}_0$

② $A, B \in \mathcal{A}_0 \Rightarrow A \cup B \in \mathcal{A}_0$

DeMorgans Law implies

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{A}_0$$

<u>σ-Algebra of sets</u>: A nonempty collecton $\mathcal{A}$ of subsets of Ω is called a σ-algebra if:

① $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

② $A_n \in \mathcal{A}$ for $n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Again DeMorgan implies

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{A}.$$

<u>Monotone Class</u>: A nonempty collection $\mathcal{D}$ is called a monotone class if

① $A_n \in \mathcal{D}$ and $A_n \subseteq A_{n+1}$ for $n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{D}$

② $A_n \in \mathcal{D}$ and $A_n \supseteq A_{n+1}$ for $n \in \mathbb{N} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{D}$

Theorem 1.1 Monotone class theorem

Let Ω be a set and $\mathcal{A}_0$ an algebra of subsets of Ω. Let $\mathcal{D}$ be collection of subsets of Ω such that $\mathcal{D} \supseteq \mathcal{A}_0$ and $\mathcal{D}$ is a monotone class. Then $\mathcal{D} \supseteq \mathcal{A}(\mathcal{A}_0)$, the σ-algebra generated by $\mathcal{A}_0$.

The proof is Monday. It should help with the last homework problem.

Homework is due next week.

$\Omega = \{1,2,3\}$, $\mathcal{A} = \{\{1\},\{2\},\{1,2\},\{3\},\{1,3\},\{2,3\},\{1,2,3\},\emptyset\}$

is an example of an algebra. Is this also a σ-algebra?