Math 713 Summary Sept 1

Define $\mathcal{P}(\Omega) = \{ A : A \subseteq \Omega \}$.

Claim $\mathcal{P}(\Omega)$ is a σ-algebra.

① If $A \in \mathcal{P}(\Omega)$ then $A^c = \Omega \setminus A \subseteq \Omega$ and so $A^c \in \mathcal{P}(\Omega)$

② If $A_i \in \mathcal{P}(\Omega)$ then $\bigcup_{i=1}^{\infty} A_i \subseteq \Omega$ and so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{P}(\Omega)$.

Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras, then $\mathcal{A}_1 \cap \mathcal{A}_2$ is also a σ-algebra.

① Let $A \in \mathcal{A}_1 \cap \mathcal{A}_2$ then $A \in \mathcal{A}_1$ and $A \in \mathcal{A}_2$.
Since $A \in \mathcal{A}_1$ and $\mathcal{A}_1$ is a σ-algebra then $A^c \in \mathcal{A}_1$
—"— $\mathcal{A}_2$ " $\mathcal{A}_2$ ——"—— $\mathcal{A}_2$

Therefore $A^c \in \mathcal{A}_1 \cap \mathcal{A}_1$.

② Let $A_i \in \mathcal{A}_1 \cap \mathcal{A}_2$ for all i. Then $A_i \in \mathcal{A}_1$ and so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_1$. Further $A_i \in \mathcal{A}_2$ implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_2$.
It follows that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_1 \cap \mathcal{A}_2$.

Suppose H is a collection of σ-algebras. Then $\bigcap_{\mathcal{A} \in \mathcal{C}} \mathcal{A}$ is a σ-algebra. The proof is the same. For example if $A \in \bigcap_{\mathcal{A} \in H} \mathcal{A}$ then $A \in \mathcal{A}$ for every $\mathcal{A} \in H$. It follows that $A^c \in \mathcal{A}$ for every $\mathcal{A} \in H$ and therefore $A^c \in \bigcap_{\mathcal{A} \in H} \mathcal{A}$.

Given $\mathcal{A}_0$ a collection of subsets of Ω define $\mathcal{A}(\mathcal{A}_0)$ the σ-algebra generated by $\mathcal{A}_0$ to be

$$\mathcal{A}(\mathcal{A}_0) = \bigcap_{\mathcal{A} \in H} \mathcal{A}$$

where $H = \left\{ \mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } \Omega \text{ with } \mathcal{A}_0 \subseteq \mathcal{A} \right\}$

Since $\mathcal{P}(\Omega) \in H$ then H is non-empty and $\mathcal{A}(\mathcal{A}_0)$ is well defined.

Note that $\mathcal{A}(\mathcal{A}_0)$ is the smallest σ-algebra which contains $\mathcal{A}_0$. In particular if $\mathcal{A}$ is a σ-algebra that contains $\mathcal{A}_0$, then $\mathcal{A} \in H$ and so $\mathcal{A} \supseteq \bigcap_{\mathcal{A} \in H} \mathcal{A} = \mathcal{A}(\mathcal{A}_0)$.

The same construction can be done with monotone classes. In particular

$$\mathcal{F} = \bigcap_{\mathcal{D} \in G} \mathcal{D}$$

where

$$G = \{ \mathcal{D} : \mathcal{D} \text{ is a monotone class and } \mathcal{A}_0 \subseteq \mathcal{D} \}$$

defines the smallest monotone class that contains $\mathcal{A}_0$. Since every σ-algebra is a monotone class then $H \subseteq G$ and therefore $\mathcal{F} \subseteq \mathcal{A}(\mathcal{A}_0)$.

The monotone class theorem says that $\mathcal{F} = \mathcal{A}(\mathcal{A}_0)$ when $\mathcal{A}_0$ is an algebra. In particular

<u>Monotone class theorem</u>: Let Ω be a set and $\mathcal{A}_0$ an algebra of subsets of Ω. Let $\mathcal{D}$ be a collection of subsets of Ω such that $\mathcal{D} \supseteq \mathcal{A}_0$ and $\mathcal{D}$ is a monotone class. Then $\mathcal{D} \supseteq \mathcal{A}(\mathcal{A}_0)$.

<u>Proof</u>: Let $\mathcal{F}$ be the smallest monotone class that contains $\mathcal{A}_0$.

Claim $\mathcal{F}$ is an algebra.

If we prove this claim, then it would follow from HW#6 that $\mathcal{F}$ is a σ-algebra.

In this case $\mathcal{F} = \mathcal{A}(\mathcal{A}_0)$ since $\mathcal{A}(\mathcal{A}_0)$ is the smallest σ-algebra containing $\mathcal{A}_0$. Now, for any monotone class $\mathcal{D}$ such that $\mathcal{D} \supseteq \mathcal{A}_0$ we have $\mathcal{F} \subseteq \mathcal{D}$ since $\mathcal{F}$ is the smallest monotone class which contains $\mathcal{A}_0$. Thus $\mathcal{A}(\mathcal{A}_0) \subseteq \mathcal{D}$.

All that remains is to prove that $\mathcal{F}$ is an algebra. To do this we need to show.

① $F, G \in \mathcal{F}$ implies $F \cup G \in \mathcal{F}$

② $F \in \mathcal{F}$ implies $F^c \in \mathcal{F}$.

Before proving ① we prove the half way step

$\frac{1}{2}$ $A \in \mathcal{A}_0$, $F \in \mathcal{F}$ implies $A \cup F \in \mathcal{F}$.

Let $A \in \mathcal{A}_0$ and define

$$\mathcal{E} = \{F \in \mathcal{F} : A \cup F \in \mathcal{F}\}$$

It is enough to show that $\mathcal{E} = \mathcal{F}$ to prove claim $\frac{1}{2}$.

First note that $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{A}_0 \subseteq \mathcal{E}$. To see that $\mathcal{A}_0 \subseteq \mathcal{E}$ let $F \in \mathcal{A}_0$. Then $A \cup F \in \mathcal{A}_0 \subseteq \mathcal{F}$ since $\mathcal{A}_0$ is an algebra. It follows that $F \in \mathcal{E}$. Thus $\mathcal{A}_0 \subseteq \mathcal{E}$.

Claim that $\mathcal{E}$ is a monotone class. If we show this then $\mathcal{E} = \mathcal{F}$ since $\mathcal{F}$ is the smallest

To show $\mathcal{E}$ is a monotone class we need to show it is closed under monotone intersections and unions, Let $F_i \in \mathcal{E}$ where $F_i \subseteq F_{i+1}$ for $i \in \mathbb{N}$. Then $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$ since $F_i \in \mathcal{F}$ and $\mathcal{F}$ is a monotone class. Moreover, $F_i \in \mathcal{E}$ implies $A \cup F_i \in \mathcal{F}$ where $A \cup F_i \subseteq A \cup F_{i+1}$ for $i \in \mathbb{N}$. Thus $A \cup F_i$ is a monotone sequence of sets in $\mathcal{F}$ and therefore $\bigcup_{i=1}^{\infty} (A \cup F_i) \in \mathcal{F}$.

Since $\bigcup_{i=1}^{\infty} (A \cup F_i) = A \cup \left(\bigcup_{i=1}^{\infty} F_i \right)$

then $F = \bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$ and $A \cup F \in \mathcal{F}$.

This means $F \in \mathcal{E}$. Thus $\bigcup_{i=1}^{\infty} F_i \in \mathcal{E}$.

Similarly if $F_i \in \mathcal{E}$ where $F_i \supseteq F_{i+1}$ for $i \in \mathbb{N}$ then $\bigcap_{i=1}^{\infty} F_i \in \mathcal{E}$. It follows that $\mathcal{E}$ is a monotone class.

We have now proved claim $\frac{1}{2}$. To prove claim 1 let $F \in \mathcal{F}$ and define

$$\mathcal{G} = \{ G \in \mathcal{F} : G \cup F \in \mathcal{F} \}$$

Claim $\mathcal{G}$ is a monotone class which contains $\mathcal{A}_0$.

To see that $\mathcal{A}_0 \subseteq \mathcal{G}$ let $G \in \mathcal{A}_0$. Then $G \cup F \in \mathcal{F}$ by claim $\frac{1}{2}$. It follows that $G \in \mathcal{G}$. Thus $\mathcal{A}_0 \subseteq \mathcal{G}$.

Let $G_i \in \mathcal{G}$ where $G_i \subseteq G_{i+1}$ for $i \in \mathbb{N}$. Then $\bigcup_{i=1}^{\infty} G_i \in \mathcal{F}$ since $G_i \in \mathcal{F}$ and $\mathcal{F}$ is a monotone class. Moreover, $G_i \in \mathcal{G}$ implies $G_i \cup F \in \mathcal{F}$ where $G_i \cup F \subseteq G_{i+1} \cup F$ for $i \in \mathbb{N}$. Thus $G_i \cup F$ is a monotone sequence of sets in $\mathcal{F}$ an therefor $\bigcup_{i=1}^{\infty} (G_i \cup F) \in \mathcal{F}$.

Since $\bigcup_{i=1}^{\infty}(G_i \cup F) = \left(\bigcup_{i=1}^{\infty} G_i\right) \cup F \in \mathcal{F}$

Then $G = \bigcup_{i=1}^{\infty} G_i$ is in $\mathcal{F}$ and $G \cup F \in \mathcal{F}$.

This means $G \in \mathcal{G}$. Thus $\bigcup_{i=1}^{\infty} G_i \in \mathcal{G}$.

Similarly if $G_i \in \mathcal{G}$ where $G_i \supseteq G_{i+1}$ for $i \in \mathbb{N}$ then $\bigcap_{i=1}^{\infty} G_i \in \mathcal{G}$. It follows that $\mathcal{G}$ is a monotone class containing $\mathcal{A}_0$. Since $\mathcal{F}$ is the smallest monotone class containing $\mathcal{A}_0$ we have that $\mathcal{F} = \mathcal{G}$.

To prove claim ② define

$$\mathcal{H} = \{F \in \mathcal{F} : F^c \in \mathcal{F}\}.$$

Claim $\mathcal{H}$ is a monotone class that contains $\mathcal{A}_0$.

Let $F \in \mathcal{A}_0$. Then $F^c \in \mathcal{A}_0 \subseteq \mathcal{F}$ implies that $F \in \mathcal{H}$. Thus $\mathcal{A}_0 \subseteq \mathcal{H}$.

Now let $F_i \in \mathcal{H}$ with $F_i \subseteq F_{i+1}$. Then since $\mathcal{F}$ is a monotone class we have $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$.

Now $F_i \in \mathcal{H}$ implies $F_i^c \in \mathcal{F}$. Thus $F_i^c \supseteq F_{i+1}^c$ is a monotone sequence in $\mathcal{F}$ and therefore $\bigcap_{i=1}^{\infty} F_i^c \in \mathcal{F}$

Since $\bigcap_{i=1}^{\infty} F_i^c = \left(\bigcup_{i=1}^{\infty} F_i\right)^c$ by De Morgan's Laws we have

That $F = \bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$ and $F^c \in \mathcal{F}$.

It follows that $F \in \mathcal{H}$, or in other words that $\bigcup_{i=1}^{\infty} F_i \in \mathcal{H}$.

The argument to show $\bigcap_{i=1}^{\infty} F_i \in \mathcal{F}$ where $F_i \supseteq F_{i+1}$ for $i \in \mathbb{N}$ is similar.

This finishes the proof of the monotone class theorem.

<u>Question:</u> If $\mathcal{B}$ is a finite collection of subsets of Ω is it necessarily a monotone class?