Math 713 Summary Sept 8, '10

Discusion of homework problems. See the posted solutions under homework.

<u>δ-ε definitions of limit:</u>

Try to remember them and write them down.

① $\lim_{n\to\infty} x_n = L$

$\Longleftrightarrow$

$\forall \varepsilon > 0\ \exists N > 0$ s.t. $n \geq N$ implies $|x_n - L| < \varepsilon$.

② x_n is a Cauchy sequence

$\Longleftrightarrow$

$\forall \varepsilon > 0\ \exists N > 0$ s.t. $m, n \geq N$ implies $|x_n - x_m| < \varepsilon$.

③ $f_n \to f$ pointwise

$\Longleftrightarrow$

$f_n : A \to \mathbb{R}$ for all $n \in \mathbb{N}$, $f : A \to B$ and $\forall x \in A$ and $\forall \varepsilon > 0\ \exists N > 0$ s.t. $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$

Note that $N = N_{x,\varepsilon}$ is a function of x and ε.

④ $f_n \to f$ uniformly

$\Longleftrightarrow$

$\forall \varepsilon > 0\ \exists N > 0$ s.t. $\forall x \in A$. $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$.

Note that $N = N_\varepsilon$ is a funtion of only ε.

④ $\lim_{x \to a} f(x) = L$

$\Leftrightarrow$ $\forall \varepsilon > 0 \ \exists \delta > 0$ st $\forall x \in \text{Domain}(f)$
then $0 < |x-a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

⑤ f is continuous at a.

$\Leftrightarrow$ $f(a)$ is defined and
$\forall \varepsilon > 0 \ \exists \delta > 0$ st $\forall x \in \text{Domain}(f)$
then $|x-a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

⑥ f is uniformly continuous

$\Leftrightarrow$ $\forall \varepsilon > 0 \ \exists \delta > 0$ st $\forall x, y \in \text{Domain}(f)$
then $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.
Note that δ is a function of ε only whereas in definition ⑤ δ is a function of a and ε.

⑦ $\lim_{x \to \infty} x_n = \infty$

$\Leftrightarrow$

$\forall M > 0 \ \exists N > 0$ s.t. $n \geq N$ implies $x_n > M$.

⑧ $\lim_{x \to \infty} x_n = -\infty$

$\Leftrightarrow$

$\forall M > 0 \ \exists N > 0$ s.t. $n \geq N$ implies $x_n < -M$.

Theorem: Completeness of $C(A, B)$.
If $f_n : A \to B$ is continuous and $f_n \to f$ uniformly then f is continuous.

The idea is to use the triangle inequality to introduce intermediate points of comparison.

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + \underbrace{|f_n(x) - f_n(a)|}_{} + |f_n(a) - f(a)|$$

to make this term small we need $|x - a| < \delta$ where δ depends on which function n.

So ...

n has to be chosen before δ.

But ...

x is controlled by δ so ...

n has to be chosen independant of x. This is why the N is definition ④ for uniform convergence doesn't depend on x.

Proof of Theorem:

Let $\varepsilon > 0$. Since $f_n \to f$ uniformly $\exists N > 0$ s.t. $\forall x \in A$ $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon/3$.

Let $n = N$. Then f_n is continuous so $\exists \delta > 0$ s.t. $x \in A$ and $|x - a| < \delta$ implies that $|f_n(x) - f_n(a)| < \varepsilon/3$.

Therefore, for $|x - a| < \delta$

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

which shows f is continuous.

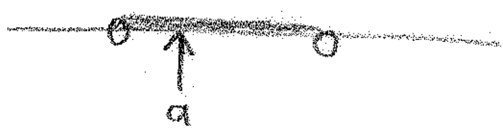
Definition of open set:

$\mathcal{O} \subseteq \mathbb{R}$ is open

$\iff$

$\forall a \in \mathcal{O}\ \exists \delta > 0$ s.t. $|x - a| < \delta$ implies $x \in \mathcal{O}$.

Example $(0,1)$ is an open set.



proof: Let $a \in (0,1)$ and choose $\delta = \min\left(\frac{a}{3}, \frac{1-a}{3}\right)$

Then $|x-a| < \delta$ implies

$$x = x - a + a \leq |x-a| + a \leq \frac{1-a}{3} + a$$
$$= \frac{1}{3} + \frac{2}{3}a < \frac{1}{3} + \frac{2}{3} = 1.$$

and

$$x = a - a + x \geq a - |x-a| \geq a - \frac{a}{3}$$
$$= \frac{2}{3}a > 0$$

Therefore $x \in (0,1)$, thus finishing the proof.

Note: there is a new homework assignment posted on the webpage. Also the are review sessions on Friday and Saturday.